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Physiology. — *The problem of the interstitial cells in the nervous endformation* [1]. By J. BOEKE, LL.D., M.D. Utrecht.

(Communicated at the meeting of February 28, 1942.)

In 1894 CAJAL described in the wall of the intestine and different glands a mesh- or network of cells, which he called "neurones sympathiques interstitiels", small triangular or spindleshaped cells having cell processes which seemed to anastomose with each other, and which stained black in his GOLGI preparations. These were found in the sympathetic nervous plexus in the wall of the stomach of frogs and different mammals, and in different glands (pancreas, salivary glands). In 1911 and 1936 he adhered to his original description, only he let fall his original doubt as to their anastomosing with each other and proclaimed them the only neurones showing a syncytial arrangement. They seemed to be connected with the smooth muscle elements ("il résulte de cette description qu'il existe dans les muscles lisses deux sortes d'arborisation nerveuses; les principales qui proviennent des grandes cellules du plexus d'AUERBACH et qui sont en même temps les plus nombreuses, et les accessoires qui émanent des cellules interstitielles". CAJAL, 1894, 1936). CAJAL did not observe a distinct connexion of his interstitial cells with the classic sympathetic elements, and he puts forwards as a very cautious hypothesis the idea that the interstitial elements are influenced by the sympathetic fibres entering the intestinal wall. In the 46 years following the first description by CAJAL these interstitial elements were studied by a number of authors; some regarded them as being of mesenchymatous nature, connective elements, others regard them as being of nervous origin, as CAJAL, LA VILLA a.o. did, but however they regarded them, they all agreed that they form a syncytium (DOGIEL, JOHNSON, COLE, etc.). LA VILLA described them in 1897 as being of nervous nature and origin, and compared them with the primitive ganglion cells of the avertebrates, in which I followed him in 1935; BETHE (1903) described a ground-net of a definitely nervous nature in the mucosa of the frog's mouth, which he homologised with the interstitial cells of CAJAL. ERIK MUELLER described the same thing in 1921, LEONTOWITSCH described a syncytial nervous groundnet in the wall of arteries containing small nerve-cells, which were identical to the interstitial cells (1927), MUENCH and SCHOCK (1905, 1910) described them as occurring in great numbers in the very loose connective tissues of the iris; their description and opinion were followed by WOLFRUM in 1931 and partly by myself in 1933 and 1936. Elements of the same nature were described after staining with methylene blue in the wall of the intestine by OKAMURA in 1935 and 1939, and by SCHABADASCH in 1934, who agreed with my descriptions and suggestions of 1933.

In 1926 LAWRENTJEW investigated these interstitial elements in my laboratory and gave them a central position in his description of the end-formation of the sympathetic nervous system. CAJAL could not find any definite connexion between these cells and the elements of the plexus of AUERBACH or MEISSNER, but supposed that they were under the influence of the real sympathetic ganglion cells. LAWRENTJEW followed up this suggestion and found them to be always lying at the end of the neurofibrillar strands of the plexus and in this way forming an intermediate element between the strands and the smooth muscle-cells.

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[1] The problem will be discussed more fully and with the necessary illustrations in the Acta Neerlandica Morphologiae, of this year (Vol. V), as XII. Innervationsstudie. There the literature bearing on the subject will be more fully accounted for.

VAN ESVELD, who followed him, explained the results of his physiological experiments (which showed him, that even after all the ganglion cells were removed, an isolated strip of the smooth muscle of the intestine still contracted rhythmically and was influenced by drugs influencing the nervous elements), by the presence of these interstitial elements inside the muscle-tissues even when the true ganglion cells had been removed. According to both workers the interstitial elements were arranged syncytially and formed the end of the efferent sympathetic pathway, being connected with the protoplasm of the innervated elements by means of an intermediate network of protoplasmic origin, the "periterminal network", described by BOEKE and by HERINGA.

In 1938 they were described very elaborately by TINEL in his excellent book on the vegetative system (MASSON, 1938), and he put them into the centre of his descriptions of the sympathetic endformation.

In 1937 they were studied as accurately as possible in my laboratory by LEEUWE, as a continuation of my own researches. In the veterinary laboratory of anatomy they have been studied very accurately by MEYLING in the wall of the aorta and in the glomus caroticum. All these authors came to the conclusion that they are nervous in nature. An intermediate position was taken by BLOOM, who admits that they are probably of nervous origin, but that they might possibly be of a microglial nature (1931). This however, is improbable, since CAJAL (and others, as LEEUWE and MEYLING) described a neurofibrillar structure in their protoplasma, which even changed its aspect in hibernating animals (CAJAL, 1911).

Thus we see that the problem is far from settled, but that the trend of the observations is in the direction of declaring them to be of nervous nature and origin but not stating their function.

LAWRENTJEW declared them to be of lemmoblastic origin, but lying at the end of the sympathetic plexus; from them the ground-bundle of neurofibrils may pass on to other interstitial elements, always maintaining a syncytial arrangement, or give off small end-knobs, the motor endings on smooth muscle-fibres. They form the real motor endings of the sympathetic plexus, and from this point of view it would be strange that they were of lemmoblastic nature. According to LEEUWE however they are true ganglion cells, which are in a syncytial connexion with the sheath-cells of the sympathetic plexus.

LEEUWE studied these elements by means of the methylene blue method, and as his work was done in my laboratory and we discussed most of his preparations and studied them together, I will describe here his conclusions more fully, because I am responsible for his work to a certain extent, and fully agree with most of his statements and descriptions, mentioned here. LEEUWE studied the interstitial elements in the enteric plexus of different animals, mammals and frogs, in embryonic tissue and in the full-grown animals, and his methods of staining enabled him to use in-toto-preparations of the thin intestinal wall of the larvae and embryoes, which was of a great advantage in the study of their embryonic development. In the submucous tissue of the frog's mouth, the region where BETHE had described his nervous groundnet, which he homologised with the interstitial elements, LEEUWE succeeded to demonstrate this network with the utmost exactness, contrary to ABRAHAM, who could not find this network (1936) and denied its existence. In the frog's intestine the structure of the enteric plexus LEEUWE found to be similar to that in mammals, and in total preparations of the intestine of frog larvae and small mammals even the development of the interstitial cells could be followed with exactness. They grow out from clusters of the ganglion cells of the developing sympathetic plexus, radiating from them as distinct elements with branching processes, but always in syncytial continuity. They thus spread out into the musculature until they reach the muscle-cells themselves. It was even possible to follow the development of the neurofibrillar structure of these syncytial elements, and it was of interest that the neurofibrillar structure, which appears in these elements, did not begin in the elements of the plexus from which the interstitial syncytium had grown out, but it showed itself first at the terminations of the elements of the end-formation just where the strands came into contact with the muscle-cells. From here it became visible passing backwards, reaching in the end the elements of the primary plexus which were already fibrillated. Thus the end-formation of the syncytial plexus of the developing interstitial elements in which neurofibrillae were visible was connected with the elements of the primary plexus, in which neurofibrillae were also visible, by a series of syncytial non-differentiated elements, which gradually became fibrillated from the periphery towards the centre.

LEEUWE could show, that the interstitial elements possessed NISSL-bodies in their protoplasm, just like ordinary ganglion cells; in the same way they showed other features of ordinary ganglion cells, for instance with regard to the effects of the oxydase- and peroxydase-reaction. The π-granules, which form characteristic elements in the protoplasm of lemmoblasts, could not be detected inside interstitial cells. Thus the so-called interstitial cells belong to the group of ganglion cells and not to that of the lemmoblasts. They are derived from the ganglion cells by a series of intermediate forms. They are always in syncytial connexion with each other and with true ganglion cells of the sympathetic plexus, and must be regarded as a kind of primitive ganglion cells. This same conclusion I had drawn from my own observations some years ago (1935, 1936), though I did not feel entitled to draw such a sharp line of separation between the neuronic elements and the so-called lemmoblasts, or sheath-cells. These too are in syncytial connexion with the ganglion cells (as I showed in the XIth Innervationsstudy, 1941, and in former publications, 1916, 1926). LEEUWE regarded all the sheath elements of the enteric plexus as interstitial elements, in which I could not follow him. There where the interstitial elements appear in the end-formation, they are no more enveloped by lemmoblasts, but they are in syncytial connexion with the lemmoblasts which surround the postganglionic fibres of the ganglion cells of the enteric plexus themselves.

Three questions have to be answered with regard to these interstitial elements: a. are all the interstitial elements lying at the end of the sympathetic endformation of an efferent nature, or is it possible, that afferent elements too belong to the system of the interstitial elements?

b. what are the definite relations of the interstitial elements to the lemmoblasts, the SCHWANN and the REMAK cells?

c. are the interstitial elements to be found exclusively in the endformation of the sympathetic system, or are elements resembling them to be found at the end of the spinal and cerebral nerves too?

Ad a. In my former papers I had drawn the conclusion, that the sympathetic groundplexus must be especially of an efferent nature ("jedenfalls ist er sicher vorwiegend efferenter Art", BOEKE, 1935). Several authors however described small ganglion cells which are undoubtedly of an interstitial nature (LEONTOWITSCH, 1921, 1930; OKAMURA, 1930, 1937; BETHE, 1903; MEYLING, 1938; SMIRNOW, 1895; DOGIEL, 1898) and of an *afferent* sensible nature, and my own observations tend in the same direction; so without doubt we have to distinguish two sorts of interstitial elements, but it seems to me that the sympathetic groundplexus is *for the greater part of an efferent nature.*

Ad b. As I mentioned before, LAWRENTJEW, who in 1926 for the first time described the interstitial cells as lying at the end of the sympathetic plexus, maintained that they formed the real endings of the sympathetic plexus, but nevertheless he identified them with the lemmoblasts ("wir sind berechtigt, die interstitiellen Zellen als Lemmoblasten zu bezeichnen". LAWRENTJEW, 1926). I agree with him fully that the interstitial elements are found lying at the end of the sympathetic plexus, and that they form real synaptic endings on the smooth muscle-fibres, giving off small end-knobs, surrounded with a distinct periterminal network inside the protoplasm of the muscle cells (cf. LEEUWE and MEYLING), but then they must be of the nature of ganglion cells and not of lemmoblasts, although they are everywhere in a syncytial connexion with the lemmoblasts (cf. LEEUWE, PEI-LIN LI, 1940, and many others). We find a whole series of intermediate forms between these elements and true ganglion cells (BOEKE, 1935, 1937; LEEUWE, 1937). These elements form real synapses (contrary to the theory of the "Terminalreticulum" of STOEHR, which in its original conception formed the connexion with the innervated elements without any trace of a synaptic junctional region where the stimulus was remoulded, polarised, a true "Umwertungsstelle der Erregung"); they are necessary for the humoral transfer of the nervous stimulus (cf. BOEKE, 1937, 1940); there may be found several interstitial elements lying one after the other at the end of the plexus, in syncytial continuity, thus forming a prolonged ending, an elongated "active region", a "plexiform innervation", but they are always of the nature of ganglion cells. Nevertheless they are always in a syncytial connexion with the lemmoblasts and in my opinion we must be very careful, before drawing such a sharp line of demarcation between the elements known as lemmoblasts and the true ganglion cells, especially where these may be of a primitive nature and liable to a very profound differentiation into different forms, and having an intermediate function, form and structure.

Ad c. Are they to be found exclusively at the end of the sympathetic endformation, as is maintained by LEEUWE and by every author who has been studying them, including CAJAL himself, LA VILLA a.s.o.?

In the sympathetic endformation they are connected with the innervated elements by means of a distinct intermediate protoplasmic structure, the periterminal network, which consists of an alveolar structure of the living substance, into which the conducting elements, the neurofibrillar structure of the terminal nervous cells, are radiating. The interstitial cells themselves show a marked alveolar structure of their protoplasma, and they may be regarded as the elements where the humoral energy is produced, the neurohumoral region of the endformation (BOEKE, 1935, 1937; LEEUWE, 1937; MEYLING, 1938). But does this hold true only and exclusively for the sympathetic endformation?

In principle the answer to this question has been given already in a lecture delivered by invitation of the Universities of London and Oxford in the year 1937, which was published with three other lectures by the Oxford Clarendon Press in 1940[1]). Perhaps I may quote here what I said at the end of this lecture: "the unsolved riddle of the cells of the core of the sensory corpuscles, the tactile cells, the elements surrounding and carrying the terminal arborisation of so many different sensory corpuscles, would perhaps find a solution through the conception of the interstitial elements. These too may perhaps be regarded as interstitial elements, intermediate elements between the conducting nerve elements and the elements receiving sensory excitations" (l.c. page 121).

As it was mentioned before, even in the endformation of the sympaticus a synapse, a junctional region, where the nervous stimulus may be remoulded, polarized, "eine Umwertungsstelle der Erregung", is necessary. It is difficult to ascribe this property and the formation of a specific hormone, necessary for the humoral transfer of the nervous stimulus, to a tiny endknob, which is present on only one in hundred smooth musclecells, as described by histologists (STOEHR, 1934). The need for a stronger mode of innervation is obvious, more adequate to fulfil the claims of modern physiology of innervation, an active region, a plexiform innervation. But not only in the sympathetic endformation this need is obvious. Even in somatic innervation, in the first place in *sensory innervation,* as soon as a sensory corpuscle has to fulfil a more complicated purpose, that is to say, that by its structure and situation it gives the impression of being able to respond to stimuli of a more delicate nature, such as may be attributed to sensory corpuscles of higher order, it shows an enlargement of the surface of its neurofibrillar structure in order to convert the most delicate impressions to active stimuli. In this way the complicated neurofibrillar structures of the so very delicately sensitive corpuscles of MEISSNER, of KRAUSE, RUFFINI, GRANDRY or HERBST may find an explanation. In the endbulbs of KRAUSE for example the whole mass of convolutions of the terminal neurofibrillae inside the bulb may be compared with the "active stretch" of synaptic value. A knob-like ending is not found here, but an "active region", a "wirksame Strecke" of the neurofibrillar endformation, and the same holds true for the MEISSNER corpuscles with their complex neurofibrillar structure, its convolutions with large ribbon-like flattened expansions, gradually breaking up into numerous thin twigs, forming most complicated loops and twists, which are everywhere surrounded by a periterminal network, lying within the protoplasm of the flattened wedge-shaped cells of the core and in continuous connexion with the neurofibrillar endloops. In the corpuscles of GRANDRY the neurofibrillar structure is moulded into a flat disc, but even here this disc is in continuous syncytial connexion with the periterminal network of the protoplasm of the two surrounding tactile cells. The same holds true for the endbulbs of KRAUSE, and for the lamellated corpuscles, as was described already some years ago (2nd Innervations-study, 1933).

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[1]) J. BOEKE, Problems of nervous anatomy, Oxford, University Press, 1940, 3rd chapter.

Here too the base of the structure called the periterminal network is formed by an alveolar structure of the protoplasm, which indicates a secretory function, the location there of the neuro-humoral region, necessary for the transfer of the nervous stimulus. This is for example emphasized by the mobility of the nucleus of the tactile cells of the corpuscles of GRANDRY following the stimulus and the changes of the mitochondrial apparatus of these cells during the nervous stimulation (SZYMONOWICZ, BOEKE, DIJKSTRA).

The cells of the core of the sensory corpuscles, the tactile cells, connected syncytially with the neurofibrillar structure of the nervous endformation, must be of the nature of a *receptor* of the nervous stimulus. They must have therefore another function than the lemmoblasts, which only conduct the nervous impulse, with which they are however in a true syncytial connexion. We may regard them as the neuro-humoral region of the endformation, receiving the nervous stimulus, in close connexion with the neurofibrillar apparatus of the nerve-endings, that is to say as the elements of the cerebro-spinal nervous endformation, to which we may ascribe the same function as to the interstitial elements, and however different and changed their form and aspect may be, we must regard them as homologous to the interstitial elements of the sympathetic endformation.

I need not to emphasize here the fact, that such a conception is entirely incompatible with the classical doctrine of the neurone theory, according to which the neurones are and remain independant units without a single syncytial stage either in development or in their fullgrown state.

But if these elements of the core of the sensory corpuscles are to be compared with the interstitial elements of the sympathetic endformation, we may ask, whether it would be possible to analyse even the motor endings, the motor endplates, in this direction. Would it be possible to view even them from the same standpoint?

As is well known, the motor endplate on the cross-striated muscle-fibres is lying hypolemmally, imbedded in the sarcoplasm of the sole-plate, which forms an intermediate structure, the periterminal network, between the neurofibrillar structure of the nerve-ending and the cross-striated myofibrillae themselves. Inside this sarcoplasma are lying three kinds of nuclei, the nuclei of the sarcoplasm itself, large loosely-built nuclei, identical with the other nuclei of the muscle-fibre lying dispersed in the sarcoplasma (fundamental nuclei, noyaux fondamentaux de RANVIER), small darkly-stained nuclei accompanying the nervous arborisations (nuclei of SCHWANN, noyaux de l'arborisation de RANVIER) and nuclei of the sarcolemma, belonging to the sheath of HENLE, lying outside the sarcolemma (noyaux vaginaux de RANVIER). In the developing motor endplates we can state that the small nuclei accompanying the nervous arborisations, are in reality derived from the ingrowing nerve-fibres and identical with their nuclei.

Now it is interesting to note, that as long ago as 1909 and 1910 THULIN and HOLMGREN described what they called "the interstitial cells" of the cross-striated muscle-fibres, but in 1926 it was shown by NOËL, that the so-called sarcosomytes (interstitial cells) of THULIN and HOLMGREN were nothing else than the nuclei of the muscle-fibre lying directly under the sarcolemma, and I agree entirely with this description of NOËL. The interstitial cells of THULIN and HOLMGREN do not exist.

But at the other side there are some facts of the development of the motor endings and of the motor endplates on the cross-striated muscle-fibres, which together with the fact, that the small nuclei accompanying the nervous arborisation of the ingrowing motor nerve-ending are derived from these ingrowing nerve fibres, point in the same direction.

In the first place it is of interest to note that when we study the development of common striated muscle-fibres in the tongue we find the intercalation of a true syncytial stage in their innervation between the first outgrowing process of the motor nerves and the formation of true independent motor end-plates, an initial plexiform innervation and from this the definite individual innervation arises. During this process there is formed on the surface of the muscle-fibre, before a true sarcolemma is developed, a mass of nuclei surrounded by protoplasm, from which the motor end-plate, viz. its sole-plate with the nuclei arises. Nuclei of the ingrowing nerve-endfibres take part in this formation, and become the nuclei of the arborisation, exactly as the cells of the core of the sensory corpuscles, the tactile cells. In the longer muscle-fibres of the muscles of the extremities (according to TELLO, who studied these developing fibres in 1917) we find at the top of the ingrowing nerve-fibres always one or two nuclei, which do not belong to the muscle-fibre, but to the ingrowing nerve-fibres themselves (cf. TELLO, 1917, fig. 33, page 173). Where a nucleus is to be seen, it must be surrounded by protoplasm [1]) (which in the preparations of TELLO was not stained and therefore invisible), and we see how this mass of protoplasm and nuclei is incorporated in the sarcoplasma of the growing sole-plate. From these nuclei are derived the nuclei of the arborisation, and in my opinion these nuclei with their surrounding protoplasma are homologous with the syncytial cells of the core of the sensory corpuscles and with the interstitial elements of the sympathetic endplexus. They form together with the sarcoplasma of the sole-plate and its nuclei (noyaux fondamentaux de RANVIER) the neuro-humoral region of the nerve-ending. in which the periterminal network is formed and in which the humoral energy is produced, which is necessary for the transmission of the nervous stimulus.

Thus it seems to me that there is every reason to suppose that we have to do here with the selfsame elements. In evertebrates they form the sympathetic ganglion cells, bearing the neuroplasm and taking over or giving off the nervous impulses. In amphioxus they still appear as anastomosing elements, primitive ganglion cells, which are in connexion with the smooth muscle cells of the intestine (BOEKE, 1935), and they often show their connection with them in the form of the dendritlamellae, demonstrated in the sympathetic ganglion cells of the vertebrates by LAWRENTJEW, and which form in Amphioxus the common mode of motor ending (BOEKE, Anat. Anz. Vol. XXXIII., 1908). In the vertebrates they appear as the interstitial cells of the sympathetic endformation, anastomosing with each other and with the true ganglion cells, which are here superimposed on them as controlling elements in the course of evolution. They are in a syncytial arrangement with each other and with the conducting nerve fibres and give off the terminal synapses [2]). In the peripheral synapses of the cerebro-spinal nerves, the sensory corpuscles and the motor endings they are still further specialised; here they are no more recognisable as ganglion cells. In the sensory corpuscles they form the elements of the core or the tactile cells, syncytially arranged and in living connexion with the lemmoblasts, the conducting elements, and with the neurofibrillar structure of the nerve termination; they form the neuro-humoral region of the nerve terminations, in which the nervous stimulus is transformed and the humoral energy is produced. In the motor endings, in which the motor end-plate is lying hypolemmally inside the sarcoplasma, they are entirely irrecognizable as distinct elements, and appear simply as the nuclei of the arborisation inside the sarcoplasma of the sole-plate, surrounded by their protoplasma, showing the periterminal network, the "receptive substance" of LANGLEY, and in continuous connexion with the sarcoplasma of the sole-plate of the muscle fibre itself. Here too they transform the nervous stimulus and produce the necessary humoral energy, even here building up the neuro-humoral region of the terminal formation.

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[1]) In the more elaborate paper, which is to be published in the Acta Neerlandica Morphologiae as the XII. Innervationsstudie, these observations will be discussed more fully and with illustrations.

[2]) In this connexion it is interesting to note, that even in 1895 BERKLEY suggested this, and that CAJAL sometimes speaks of "la cellule interstitielle ou terminale".

I need not to emphasize here the fact, that this conception is entirely irreconcilable with the classic neurone theory and that the identification of these interstitial elements with ganglion cells applies exclusively to their function as bearers of the neuroplasma and not to their form.

Utrecht, February 1942.

Mathematics. — *Ueber die $M_3{}^3$ dreier Ebenen im R_5.* Von R. WEITZENBÖCK.

(Communicated at the meeting of February 28, 1942.)

Drei Ebenen E_1, E_2 und E_3 im fünfdimensionalen projektiven Raume R_5 spannen eine dreidimensionale Punktmannigfaltigkeit dritten Grades $M_3{}^3$ auf, die der Klasse der sogen. SEGRE'schen Mannigfaltigkeiten angehört [1]). Für die projektive Geometrie dieser $M_3{}^3$ bildet die Theorie der binär-ternären Bilinearform die natürliche Grundlage und wird also beherrscht durch die Theorie der Punktreihen der Ebene, die durch E. A. WEISS ausführlich dargestellt wurde [2]).

Im Bereiche der senären Formen, wenn die $M_3{}^3$ durch die drei Ebenen a, α, p (oder 1, 2, 3) gegeben ist, entsteht die Frage nach jenen projektiven Komitanten dieser drei Ebenen, die, gleich Null gesetzt, als „Gleichung der $M_3{}^3$" bezeichnet werden können.

Drei Ebenen im R_5 besitzen keine projektiven Invarianten [3]) und haben auch, wie ich unlängst bewiesen habe [4]), keine Komitanten mit nur einer Reihe Punktkoordinaten x oder nur einer Reihe R_4-Koordinaten u'. Es ist daher nicht möglich die $M_3{}^3$ durch eine einzige Gleichung in x_i bzw. in u'_i darzustellen. Dagegen ist dies möglich mit einer Reihe Linienkoordinaten π_{ik} (dual mit einer Reihe R_3-Koordinaten π'_{ik}) und auch mit einer Reihe Ebenenkoordinaten π_{ijk}. Die dabei auftretenden Komitanten sollen hier ermittelt werden.

§ 1.

Am einfachsten kommt man bei Ebenenkoordinaten zum Ziel. Die allgemeine erzeugende Ebene der $M_3{}^3$ wird nämlich gegeben durch

$$E_\delta = E_1\,A_{23} + 9\delta\,J_{\pi 123} - 9\delta^2\,J_{\pi 213} + \delta^3\,E_2\,A_{31} = 0 \quad . \quad . \quad . \quad (1)$$

Dabei bedeuten:

$$\begin{aligned} A_{23} &= (2^3\,3^3) = (\alpha^3\,p^3) \\ E_1 &= (1^3\,\pi^3) = (a^3\,\pi^3) \end{aligned} \qquad J_{\pi 123} = (\pi^3\,1^2\,2)\,(1 2^2\,3^3)$$

und δ ist das Doppelverhältnis der vier Punkte in denen eine erzeugende Gerade von den Ebenen E_1, E_2, E_δ und E_3 getroffen wird [5]).

Bei Benützung der in allen drei Ebenen symmetrischen Komitante

$$S = \tfrac{1}{24}\,\Sigma\,J_{\pi 123}$$

kann man (1) auch so schreiben:

$$\left.\begin{aligned} E_\delta = E_1\,A_{23} + \delta\,(-\tfrac{3}{2}\,E_1\,A_{23} + \tfrac{1}{2}\,E_2\,A_{31} - \tfrac{1}{2}\,E_3\,A_{12} + 9S) + \\ + \delta^2\,(\tfrac{1}{2}\,E_1\,A_{23} - \tfrac{3}{2}\,E_2\,A_{31} - \tfrac{1}{2}\,E_3\,A_{12} - 9S) + \delta^3\,E_2\,A_{31} = 0 \end{aligned}\right\} \quad . \quad (2)$$

Ist in (1) oder (2) π_{ijk} gegeben, so sind die drei Wurzeln δ die Doppelverhältnisse, die auf den drei Transversalen der vier Ebenen E_1, E_2, π_{ijk} und E_3 bestimmt werden. Die drei Schnittpunkte dieser Transversalen mit der Ebene π_{ijk} sind dann die Treffpunkte dieser Ebene mit der $M_3{}^3$. Zwei dieser Punkte fallen zusammen, wenn die Diskriminante von (2)

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[1]) C. SEGRE, Rendic. di Palermo **5**, 192—204 (1891) und Mathem. Enzyklopädie III C 7, 828.

[2]) E. A. WEISS, Punktreihengeometrie, Teubner (1939). Insbes. S. 45 ff.

[3]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 1026—1029 (1932).

[4]) Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 139—141 (1942).

[5]) Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 907—913 (1941).

$$D(E_1, E_2, E_3, S) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

ist. Sie wird vom vierten Grad in den π_{ijk} und (3) *kann man als Gleichung der* $M_3{}^3$ *in Ebenenkoordinaten beschauen.*

§ 2.

Die Gleichung der $M_3{}^3$ in Linienkoordinaten $\pi_{ik} = \varrho_{ik} = \sigma_{ik}$ muss ausdrücken, dass die Gerade π_{ik} mit der $M_3{}^3$ wenigstens einen Punkt gemein hat. Wir erhalten die entsprechende Komitante wie folgt.

Durch π_{ik} und E_1 geht ein R_4, der E_2 nach einer Geraden g_{21} schneidet. Analog erhält man in E_3 eine Gerade g_{32} und in E_1 eine Gerade g_{13}. Trifft π_{ik} die $M_3{}^3$ in P, so geht durch P eine erzeugende Gerade, die g_{21}, g_{32} und g_{13} trifft, d.h. g_{13} schneidet den durch g_{21} und g_{32} bestimmten R_3. Dies gibt

$$\Omega = (\pi^2\, 1^3\, \alpha)\, (\varrho^2\, 2^3\, p)\, (\sigma^2\, 3^3\, \mathrm{a})\, (\alpha^2\, p^2\, \mathrm{a}^2) = 0 \quad . \; . \; . \; . \quad (4)$$

Die $M_3{}^3$ *wird also durch einen Linienkomplex dritten Grades dargestellt.*

Setzt man in (4) $\pi_{ik} = (xy)_{ik}$, so erhält man in laufenden Koordinaten x_i die Gleichung des Kegels dritter Ordnung, der sich ergibt, wenn man den Punkt y mit allen Punkten der $M_3{}^3$ verbindet. Nimmt man in (4) die π_{ik} als gegeben, dagegen z.B. die a_{ijk} als veränderlich, so erhält man einen quadratischen Ebenenkomplex. Er stellt die Regelschaar dar, die von allen Geraden gebildet wird, die π_{ik} und die beiden Ebenen E_2 und E_3 treffen.

Dual zu (4) erhält man

$$\Omega' = (\pi'^2\, 1'^3\, \alpha')\, (\varrho'^2\, 2'^3\, p')\, (\sigma'^2\, 3'^3\, \mathrm{a}')\, (\alpha'^2\, p'^2\, \mathrm{a}'^2) = 0 \; . \quad . \; . \; . \quad (5)$$

als *Gleichung der* $M_3{}^3$ *in* R_3*-Koordinaten* $\pi'_{ik} = \pi_{rst}$.

Mathematics. — *A remarkable family.* By J. G. VAN DER CORPUT.

(Communicated at the meeting of February 28, 1942.)

In the preceding part of this chapter[1]) I have confined myself to functions $f_\nu(x)$ of one variable, but now I consider functions $f_\nu(x_\tau)$ of t variables $x_1, \dots, x_t$, where $t \geqq 1$; in this chapter τ and ω run through the values $1, 2, \dots, t$. As in the above argument $\varkappa$ runs through the values $1, 2, \dots, k$, while μ runs through $1, 2, \dots, k-1$; finally ν and ϱ run through the values $1, 2, \dots, n$. I consider n functions $g_\varrho(x_\tau, y_{\varkappa\nu})$ of $t + kn$ variables $x_\tau, y_{\varkappa\nu}$, that are analytical at the origin $x_\tau = y_{\varkappa\nu} = 0$ of the $(t + kn)$-dimensional espace and assume at that point the value zero. Further I consider kt functions $l_{\varkappa\omega}(x_\tau)$ of the t variables $x_1, \dots, x_t$, that are analytical at the origin $x_\tau = 0$ and take at that point the value zero. Ultimately I consider the functional system

$$g_\varrho\{x_\tau, f_\nu(l_{\varkappa\omega}(x_\tau))\} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

consisting of n functional equations and involving n unknown functions $f_\nu(x_\omega)$ of the t variables $x_1, \dots, x_t$.

By $\triangle$ I denote the determinant of n rows and columns, in which the constituent in the ϱ^{th} row and ν^{th} column has the value $\left(\dfrac{\partial g_\varrho}{\partial y_{k\nu}}\right)_0$, where the suffix 0 sigmifies that we must take $x_\tau = y_{\varkappa\nu} = 0$. By D I denote the determinant of t rows and columns, in which the constituent in the ω^{th} row and τ^{th} column is $\left(\dfrac{\partial l_{k\omega}}{\partial x_\tau}\right)_0$.

Speaking of a system of degree α I mean a system of t integers $\geqq 0$, the sum of which is α. The number of systems of degree α is $\binom{\alpha + t - 1}{t - 1}$ for every integer $\alpha > 0$.

If $(\eta_1, \dots, \eta_t)$ is a system of positive degree α, I can write

$$\sum_{\varkappa} \left(\frac{\partial g_\varrho}{\partial y_{\varkappa\nu}}\right)_0 \prod_{\omega} \left(\sum_{\tau} \left(\frac{\partial l_{\varkappa\omega}}{\partial x_\tau}\right)_0 x_\tau\right)^{\eta_\omega} = \sum_{\zeta_\tau} T_{\nu\varrho}(\eta_\omega, \zeta_\tau) \prod_{\tau} x_\tau^{\zeta_\tau},$$

where $\sum_{\zeta_\tau}$ is a sum over the systems $(\zeta_1, \dots, \zeta_t)$ of degree α. If ϱ runs through the values $1, 2, \dots, n$, we find so a row of $\binom{\alpha + t - 1}{t - 1} n$ numbers $T_{\nu\varrho}(\eta_\omega, \zeta_\tau)$. Further, if ν runs through the values $1, 2, \dots, n$ and $(\eta_1, \dots, \eta_t)$ through the systems of degree α, we obtain in this manner $\binom{\alpha + t - 1}{t - 1} n$ rows, that form a determinant, say D_α.

In the special case $t = 1$ we have $D = l_k'(0)$ and D_α is the above determinant of n rows and columns, in which the constituent in the ϱ^{th} row and ν^{th} column is

$$\sum_{\varkappa} \left(\frac{\partial g_\varrho}{\partial y_{\varkappa\nu}}\right)_0 (l_\varkappa'(0))^\alpha.$$

I say that a function $w(x_\tau)$ possesses at the origin $x_\tau = 0$ a zero with multiplicity $> \Omega$, if $w(x_\tau)$ takes at that point the value zero and $r^{-\Omega} w(x_\tau)$, where $r = \sqrt{\sum_\tau |x_\tau|^2}$ tends with the positive number r to zero. I say that $w(x_\tau)$ is in the vicinity of the origin $x_\tau = 0$ approximatively equal to a polynomial $p(x_\tau)$ of degree Ω, if $w(x_\tau) = p(x_\tau)$ at the origin $x_\tau = 0$ and $r^{-\Omega}\{w(x_\tau) - p(x_\tau)\}$ tends with the positive number r to zero.

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[1]) This communication is a continuation of the first part of Chapter III, published in these Proceedings, **45** (1942), p. 129—135. The chapters I and II appear in Euclides; Compare Euclides **18** (1941—'42), p. 50—78.

Theorem 2. *Suppose that the* kt ($k \geqq 2$, $t \geqq 1$) *given functions* $l_{\chi\omega}(x_\tau)$ *of* t *variables* $x_1, \dots, x_t$ *are analytical at the origin* $x_\tau = 0$ *and assume at that point the value zero, further that the* n *given functions* $g_\varrho(x_\tau, y_{\chi\nu})$ *of* $t + kn$ *variables* $x_\tau, y_{\chi\nu}$ *are analytical at the origin* $x_\tau = y_{\chi\nu} = 0$ *and take at that point the value zero, moreover that* $\triangle \neq 0$ *and* $D \neq 0$, *finally that the* $(k-1)\,t$ *inequalities*

$$\left|\sum_{\tau}\left(\frac{\partial l_{\mu\omega}}{\partial x_\tau}\right)_0 X_\tau\right| < 1 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (11)$$

are true for any system $(X_1, \dots, X_t)$ *satisfying the* t *relations*

$$\left|\sum_{\tau}\left(\frac{\partial l_{k\omega}}{\partial x_\tau}\right)_0 X_\tau\right| = 1 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (12)$$

Then there exists an integer $\Omega \geqq 0$ *such that* $D_\alpha \neq 0$ *for* $\alpha > \Omega$.

If $\Omega = 0$ (*in other words: if* $D_\alpha \neq 0$ *for* $\alpha = 1, 2, \dots$), *the functional system* (10) *possesses one and only one solution analytical and vanishing at the origin* $x_\tau = 0$.

If $\Omega > 0$ *and* $p_\nu(x_\tau)$ *denotes a polynomial of degree* Ω *such that* $p_\nu(x_\tau) = 0$ *at* $x_\tau = 0$ *and that each of the* n *functions* $g_\varrho\{x_\tau, p_\nu(l_{\chi\omega}(x_\tau))\}$ *possesses at the origin* $x_\tau = 0$ *a zero with multiplicity* $> \Omega$, *then there exists one and only solution* $(f_\nu(x_\tau))$ *of the considered functional system with the property that* $f_\nu(x_\tau)$ *is analytical at the origin* $x_\tau = 0$ *and is in the vicinity of that point approximatively equal to* $p_\nu(x_\tau)$[1]).

If $D_\alpha = 0$ *for at least one positive integer* α, *then the number of solutions analytical and vanishing at the origin* $x_\tau = 0$ *is either zero or infinite.*

The special case $t = 1$, $\Omega = 0$ of this theorem gives the first proposition.

The proof which shows a great analogy to that of the previous theorem runs as follows.

I. Recurrent relations between the coefficients.

By hypothesis there exists a $(t + kn)$-dimensional vicinity V of the origin $x_\tau = y_{\chi\nu} = 0$ with the property that the n functions $g_\varrho(x_\tau, y_{\chi\nu})$ are defined and analytical in V. The determinant $\triangle$ being $\neq 0$, there exists a $(t + (k-1)\,n)$-dimensional vicinity V_1 of the point $x_\tau = y_{\mu\nu} = 0$ such that n analytical functions $h_\varrho(x_\tau, y_{\mu\nu})$ can be found in V_1 with the following properties:

1. If $(x_\tau, y_{\mu\nu})$ is an arbitrary point of V_1 and we put $y_{k\varrho} = h_\varrho(x_\tau, y_{\mu\nu})$, then the point $(x_\tau, y_{\chi\nu})$ lies in V and satisfies the n relations $g_\varrho(x_\tau, y_{\chi\nu}) = 0$.
2. The n functions $h_\varrho(x_\tau, y_{\mu\nu})$ assume at $x_\tau = y_{\mu\nu} = 0$ the value zero.
3. The n analytical functions $h_\varrho(x_\tau, y_{\mu\nu})$ are defined unambiguously in V_1 by the properties 1. and 2.

The given functional system is in the vicinity of the origin $x_\tau = 0$ equivalent to

$$f_\varrho(l_{k\omega}(x_\tau)) = h_\varrho\{x_\tau, f_\nu(l_{\mu\omega}(x_\tau))\},$$

in other words: any system of n analytical functions $f_\nu(x_\tau)$ vanishing at the origin $x_\tau = 0$ and satisfying in the vicinity of that point one of both functional systems, satisfies also the other in the neighbourhood of that point.

In this proof $\tau, \varphi, \psi, \chi, \omega$ run through the values $1, 2, \dots, t$. Since $D \neq 0$, the substitution

$$l_{k\omega}(x_\tau) = z_\omega \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (13)$$

gives in the vicinity of the origin $x_\tau = 0$ an analytical (1, 1)-transformation. Hence

$$x_\tau = q_\tau(z_\omega) \text{ and } l_{\mu\omega}(x_\tau) = w_{\mu\omega}(z_\psi) \quad . \ . \ . \ . \ . \ . \quad (14)$$

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[1]) The following remark is obvious: If Ω is positive and $(f_\nu(x_\tau))$ is a solution of the considered functional system with the properties that $f_\nu(x_\tau) = 0$ at the origin $x_\tau = 0$ and that $f_\nu(x_\tau)$ is in the vicinity of that point approximatively equal to a polynomial $p_\nu(x_\tau)$ of degree Ω, then each of the n functions $g_\varrho\{x_\tau, p_\nu(l_{\chi\omega}(x_\tau))\}$ possesses at the origin $x_\tau = 0$ a zero with multiplicity $> \Omega$.

are analytical functions of $z_1, \ldots, z_t$ at the origin $z_\omega = 0$ and the functional system reduces to

$$f_\varrho(z_\omega) = h_\varrho \{ q_\varphi(z_\omega),\ f_\nu(w_{\mu\chi}(z_\psi)) \}.$$

We can write

$$w_{\mu\chi}(z_\psi) = \sum_{\beta_\psi} W_{\mu\chi}(\beta_\psi) \prod_\psi z_\psi^{\beta_\psi}, \qquad q_\varphi(z_\omega) = \sum_{\gamma_\omega} Q_\varphi(\gamma_\omega) \prod_\omega z_\omega^{\gamma_\omega}$$

and

$$h_\varrho(x_\tau, y_{\mu\nu}) = \sum_{\delta_\varphi, \varepsilon_{\mu\nu}} H_\varrho(\delta_\varphi, \varepsilon_{\mu\nu}) \prod_{\varphi,\mu,\nu} x_\varphi^{\delta_\varphi} y_{\mu\nu}^{\varepsilon_{\mu\nu}},$$

where $\beta_\psi, \gamma_\omega, \delta_\varphi$ and $\varepsilon_{\mu\nu}$ run through the sequence of integers $\geqq 0$ with the properties

$$\sum_\psi \beta_\psi > 0; \qquad \sum_\omega \gamma_\omega > 0, \qquad \sum_\varphi \delta_\varphi + \sum_{\mu,\nu} \varepsilon_{\mu,\nu} > 0.$$

First, assume that the functional system possesses a solution

$$f_\nu(z_\tau) = \sum_{\alpha_\chi} F_\nu(\alpha_\chi) \prod_\chi z_\chi^{\alpha_\chi},$$

that is analytical at the origin and takes at that point the value zero; $\sum\limits_{\alpha_\chi}$ is a sum over the systems $(\alpha_1, \ldots, \alpha_t)$ consisting of t integers $\alpha_\chi \geqq 0$, the sum of which is positive. Then we have in the neighbourhood of the origin $z_\omega = 0$

$$\sum_{\eta_\omega} F_\varrho(\eta_\omega) \prod_\omega z_\omega^{\eta_\omega} = \sum_{\delta_\varphi, \varepsilon_{\mu\nu}} H_\varrho(\delta_\varphi, \varepsilon_{\mu\nu}) \prod_\varphi \{ \sum_{\gamma_\omega} Q_\varphi(\gamma_\omega) \prod_\omega z_\omega^{\gamma_\omega} \}^{\delta_\varphi} \prod_{\mu,\nu} Y_{\mu\nu}^{\varepsilon_{\mu\nu}}, \tag{15}$$

where

$$Y_{\mu\nu} = \sum_{\alpha_\chi} F_\nu(\alpha_\chi) \prod_\chi \{ \sum_{\beta_\psi} W_{\mu\chi}(\beta_\psi) \prod_\psi z_\psi^{\beta_\psi} \}^{\alpha_\chi}. \quad . \quad . \quad . \tag{16}$$

Let α be a positive integer. To find some particular terms of degree α occurring in the expansion of the right-hand side of (15), I consider a certain μ $(1 \leqq \mu \leqq k-1)$, a certain ν $(1 \leqq \nu \leqq n)$ and I take $\delta_1 = \ldots = \delta_t = 0$, $\varepsilon_{\mu\nu} = 1$, the other exponents $\varepsilon = 0$. In that case

$$H_\varrho(\delta_\varphi, \varepsilon_{\mu\nu}) = \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}} \right)_0,$$

so that we find on the right-hand side of (15) the term $\left(\dfrac{\partial h_\varrho}{\partial y_{\mu\nu}} \right)_0 Y_{\mu\nu}$. To any ψ $(1 \leqq \psi \leqq t)$ corresponds a system $(\beta_1, \ldots, \beta_t)$ with $\beta_\psi = 1$, $\beta_\omega = 0$ $(\omega \neq \psi)$ and for that system

$$W_{\mu\chi}(\beta_\psi) = \left(\frac{\partial w_{\mu\chi}}{\partial z_\psi} \right)_0,$$

so that $Y_{\mu\nu}$ contains the sum

$$\sum_{\alpha_\chi} F_\nu(\alpha_\chi) \prod_\chi \left(\sum_\psi \left(\frac{\partial w_{\mu\chi}}{\partial z_\psi} \right)_0 z_\psi \right)^{\alpha_\chi}.$$

In this manner we see that the expansion of the right-hand side of (15) contains the expression

$$J_\varrho(z_\psi) = \sum_{\mu,\nu} \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}} \right)_0 \sum_{\eta_\chi} F_\nu(\eta_\chi) \prod_\chi \left(\sum_\psi \left(\frac{\partial w_{\mu\chi}}{\partial z_\psi} \right)_0 z_\psi \right)^{\eta_\chi},$$

where $\sum\limits_{\eta_\chi}$ is a sum over the systems $(\eta_1, \ldots, \eta_t)$ of degree α.

In this manner we do not find all terms of degree α occurring in the expansion of the right-hand side of (15). By choosing all exponents $\varepsilon_{\mu\nu} = 0$, we obtain the terms that do not involve any unknown coefficient $F_\nu(\alpha_\chi)$. We find the remaining terms by choosing the systems (α_χ), (β_ψ), (γ_ω), (δ_φ) and $(\varepsilon_{\mu\nu})$ with the property that at least one of the three inequalities

$$\sum_\psi \beta_\psi \geqq 2, \qquad \sum_\varphi \delta_\varphi \geqq 1, \qquad \sum_{\mu,\nu} \varepsilon_{\mu\nu} \geqq 2$$

is satisfied, so that these terms possess a degree $\alpha > \sum_\chi \alpha_\chi$; hence each of these remaining terms is a polynomial in the numbers $F_\sigma(\alpha_\chi)$, where $\sigma = 1, 2, \ldots, n$ and $(\alpha_1, \ldots, \alpha_t)$ is a system of degree $< \alpha$.

From (15) it follows that

$$\sum_{\eta_\chi} F_\varrho(\eta_\chi) \prod_\chi Z_\chi^{\eta_\chi} - J_\varrho(Z_\psi) \qquad \Bigl(\sum_\chi \eta_\chi = \alpha\Bigr) \quad . \quad . \quad . \quad . \quad (17)$$

is a polynomial in $Z_1, \ldots, Z_t$ of degree α with the property that each of its coefficients is a polynomial in $F_\sigma(\alpha_\chi)$, where $(\alpha_1, \ldots, \alpha_t)$ runs through the systems of degree $< \alpha$. The polynomial (17) has an expansion

$$\sum_{\nu, \eta_\chi} F_\nu(\eta_\chi) \sum_{\zeta_\varphi} S_{\nu\varrho}(\eta_\chi, \zeta_\varphi) \prod_\varphi Z_\varphi^{\zeta_\varphi} \qquad \Bigl(\sum_\chi \eta_\chi = \sum_\varphi \zeta_\varphi = \alpha\Bigr),$$

where $S_{\nu\varrho}(\eta_\chi, \zeta_\varphi)$ is the coefficient of $Z_1^{\zeta_1} \ldots Z_t^{\zeta_t}$ in the expansion of

$$\varepsilon'_{\nu\varrho} \prod_\chi Z_\chi^{\eta_\chi} - \sum_\mu \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 \prod_\chi \left\{ \sum_\psi \left(\frac{\partial w_{\mu\chi}}{\partial z_\psi}\right)_0 Z_\psi \right\}^{\eta_\chi}; \quad . \quad . \quad . \quad (18)$$

$\varepsilon'_{\nu\varrho} = 1$ for $\nu = \varrho$ and $= 0$ for $\nu \neq \varrho$. Hence

$$\sum_{\nu, \eta_\chi} F_\nu(\eta_\chi) S_{\nu\varrho}(\eta_\chi, \zeta_\varphi) = u_\varrho(\zeta_\varphi) \qquad \Bigl(\sum_\chi \eta_\chi = \alpha\Bigr) \quad . \quad . \quad . \quad (19)$$

is for $\varrho = 1, 2, \ldots, n$ and for any system $(\zeta_1, \ldots, \zeta_t)$ of degree α a polynomial in the coefficients $F_\sigma(\alpha_\chi)$, where $\sigma = 1, 2, \ldots, n$ and $(\alpha_1, \ldots, \alpha_t)$ runs through the systems of degree $< \alpha$. If we replace the right-hand side of (16) by the analogous sum over the systems $(\alpha_1, \ldots, \alpha_t)$ of degree $< \alpha$, we find in the expansion of the right-hand side of (15) precisely the terms with the coefficients $u_\varrho(\zeta_\varphi)$, so that $u_\varrho(\zeta_\varphi)$ is the coefficient of $Z_1^{\zeta_1} \ldots Z_t^{\zeta_t}$ in the expansion of $h_\varrho\{q_\varphi(z_\tau), j_\nu(w_{\mu\chi}(z_\tau))\}$, where

$$j_\nu(x_\chi) = \sum_{\alpha_\chi} F_\nu(\alpha_\chi) \prod_\chi x_\chi^{\alpha_\chi} \qquad \Bigl(\sum_\chi \alpha_\chi < \alpha\Bigr). \quad . \quad . \quad . \quad . \quad (20)$$

If the coefficients $F_\sigma(\alpha_\chi)$ are already known for the systems $(\alpha_1, \ldots, \alpha_t)$ of degree $< \alpha$, the numbers $u_\varrho(\zeta_\varphi)$ are known and (19) gives $\binom{\alpha + t - 1}{t - 1} n$ linear relations between as many unknown coefficients $F_\nu(\eta_\chi)$, where $\sum_\chi \eta_\chi = \alpha$. The determinant E_α of this system of linear equations contains $\binom{\alpha + t - 1}{t - 1} n$ rows and columns; its constituents are $S_{\nu\varrho}(\eta_\chi, \zeta_\varphi)$. Let us investigate this determinant.

II. Investigation of the determinant.

To begin with I show that E_α tends to 1, if α be increased indefinitely. From $D \neq 0$ it follows that to any system $(Z_1, \ldots, Z_t)$ corresponds a system $(X_1, \ldots, X_t)$ satisfying the t relations

$$\sum_\tau \left(\frac{\partial l_{k\omega}}{\partial x_\tau}\right)_0 X_\tau = Z_\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence it follows from (13) and (14), that

$$\sum_\omega \left(\frac{\partial w_{\mu\varphi}}{\partial z_\omega}\right)_0 Z_\omega = \sum_\tau X_\tau \sum_\omega \left(\frac{\partial w_{\mu\varphi}}{\partial z_\omega}\right)_0 \left(\frac{\partial l_{k\omega}}{\partial x_\tau}\right)_0 = \sum_\tau \left(\frac{\partial l_{\mu\varphi}}{\partial x_\tau}\right)_0 X_\tau. \quad . \quad (22)$$

Since (11) holds for any system $(X_1, \dots, X_t)$ satisfying the relations (12), we find

$$\left| \sum_{\omega} \left(\frac{\partial w_{\mu\varphi}}{\partial z_\omega} \right)_0 Z_\omega \right| < 1$$

for any system $(Z_1, \dots, Z_t)$ consisting of t numbers with absolute value $= 1$. By taking Z_ω conveniently, we obtain

$$\left(\frac{\partial w_{\mu\varphi}}{\partial z_\omega} \right)_0 Z_\omega = \left| \left(\frac{\partial w_{\mu\varphi}}{\partial z_\omega} \right)_0 \right|$$

and so we find

$$\sum_{\omega} \left| \left(\frac{\partial w_{\mu\varphi}}{\partial z_\omega} \right)_0 \right| \leqq \theta < 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where θ is a convenient number independent of μ and φ. The coefficient of $Z_1^{\zeta_1} \dots Z_t^{\zeta_t}$ in the expansion of

$$\sum_{\mu} \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}} \right)_0 \prod_{\chi} \left(\sum_{\psi} \left(\frac{\partial w_{\mu\chi}}{\partial z_\psi} \right)_0 Z_\psi \right)^{\eta_\chi},$$

where $\sum\limits_{\chi} \eta_\chi = \alpha$, is therefore in absolute value

$$\leqq \sum_{\mu} \left| \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}} \right)_0 \right| \prod_{\chi} \theta^{\eta_\chi} \leqq C\,\theta^\alpha,$$

where C is independent of $\nu, \varrho, \eta_1, \dots, \eta_t$. Since $S_{\nu\varrho}(\eta_\chi, \zeta_\varphi)$ is the coefficient of $Z_1^{\zeta_1} \dots Z_t^{\zeta_t}$ in the expansion of (18), we find

$$| S_{\nu\varrho}(\eta_\chi, \zeta_\varphi) - 1 | \leqq C\,\theta^\alpha \text{ if } \nu = \varrho \text{ and } \eta_\chi = \zeta_\chi,$$

$$| S_{\nu\varrho}(\eta_\chi, \zeta_\varphi) | \leqq C\,\theta^\alpha \quad \text{in the other cases.}$$

In other words, the constituents of E_α occuring in the principal diagonal are approximatively 1, and the other constituents are approximatively zero.

We can write E_α as a sum of $1 + \binom{s}{1} + \binom{s}{2} + \dots + \binom{s}{s}$ terms, where $s = \binom{\alpha + t - 1}{t-1} n$ denotes the number of rows of E_α. The first term is the product of the numbers 1, occurring in the principal diagonal; this term has therefore the value 1. Each of the $\binom{s}{1}$ following terms is a constituent, occurring in the principal diagonal, diminished by 1 and has therefore an absolute value $\leqq C\,\theta^\alpha$. Each of the $\binom{s}{2}$ following terms is a determinant of two rows and columns of which each constituent is in absolute value $\leqq C\,\theta^\alpha$, so that these terms are in absolute value $\leqq 2!\,(C\,\theta^\alpha)^2$, and so on. Hence

$$| E_\alpha - 1 | \leqq \sum_{h=1}^{s} \binom{s}{h} h!\,(C\,\theta^\alpha)^h \leqq \sum_{h=1}^{s} (C\,s\,\theta^\alpha)^h.$$

If α is large enough,

$$C\,s\,\theta^\alpha = C \binom{\alpha + t - 1}{t - 1} n\,\theta^\alpha < \tfrac{1}{2},$$

hence

$$| E_\alpha - 1 | < 2\,C \binom{\alpha + t - 1}{t - 1} n\,\theta^\alpha < 1. \quad . \quad . \quad . \quad . \quad (24)$$

In this manner we have found that E_α tends to 1 for $\alpha \to \infty$ and that there exists an integer $\Omega \geqq 0$ with the property $E_\alpha \neq 0$ for $\alpha > \Omega$.

Let us now prove that the relations $D_\alpha = 0$ and $E_\alpha = 0$ are equivalent. From the definition of D_α and $T_{\nu\varrho}(\eta_\chi, \zeta_\varphi)$ it follows that D_α is zero if and only if the $\binom{\alpha+t-1}{t-1}n$ polynomials

$$\sum_\varrho U_\varrho \sum_\varkappa \left(\frac{\partial g_\varrho}{\partial y_{\varkappa\nu}}\right)_0 \prod_\chi \left\{ \sum_\tau \left(\frac{\partial l_{\varkappa\chi}}{\partial x_\tau}\right)_0 X_\tau \right\}^{\eta_\chi},$$

where X_τ and U_ϱ denote $t+n$ independent variables and $(\eta_1, \dots, \eta_t)$ runs through the systems of degree α, are linearly dependent. Since $y_{k\sigma} = h_\sigma(x_\tau, y_{\mu\nu})$ satisfies the system $g_\varrho(x_\tau, y_{\chi\nu}) = 0$, we have

$$\left(\frac{\partial g_\varrho}{\partial y_{\mu\nu}}\right)_0 + \sum_{\sigma=1}^{n} \left(\frac{\partial g_\varrho}{\partial y_{k\sigma}}\right)_0 \left(\frac{\partial h_\sigma}{\partial y_{\mu\nu}}\right)_0 = 0$$

and the said polynomials have according to (21) and (22) the form

$$\left.\begin{aligned} &\sum_\varrho U_\varrho \left\{ \left(\frac{\partial g_\varrho}{\partial y_{k\nu}}\right)_0 \prod_\chi Z_\chi^{\eta_\chi} - \sum_{\mu,\sigma} \left(\frac{\partial g_\varrho}{\partial y_{k\sigma}}\right)_0 \left(\frac{\partial h_\sigma}{\partial y_{\mu\nu}}\right)_0 \prod_\chi \left\{ \sum_\tau \left(\frac{\partial w_{\mu\chi}}{\partial z_\tau}\right)_0 Z_\tau \right\}^{\eta_\chi} \right\} \\ &= \sum_{\varrho,\sigma} \left(\frac{\partial g_\varrho}{\partial y_{k\sigma}}\right)_0 U_\varrho \sum_{\zeta_\varphi} S_{\nu\sigma}(\eta_\chi, \zeta_\varphi) \prod_\varphi Z_\varphi^{\zeta_\varphi} \qquad \left(\sum_\varphi \zeta_\varphi = \alpha\right) \\ &= \sum_\sigma U'_\sigma \sum_{\zeta_\varphi} S_{\nu\sigma}(\eta_\chi, \zeta_\varphi) \prod_\varphi Z_\varphi^{\zeta_\varphi}, \end{aligned}\right\} \tag{25}$$

if we put

$$U'_\sigma = \sum_\varrho \left(\frac{\partial g_\varrho}{\partial y_{k\sigma}}\right)_0 U_\varrho.$$

From $\triangle \neq 0$ it follows that $D_\alpha = 0$ if and only if the $\binom{\alpha+t-1}{t-1}n$ polynomials (25), where Z_φ and U'_σ denote $t+n$ independent variables, are linearly dependent. This is the case if and only if $E_\alpha = 0$. Hence $D_\alpha = 0$ implies $E_\alpha = 0$ and conversely.

If $\Omega = 0$ (in other words, if $D_\alpha \neq 0$ for $\alpha = 1, 2, \dots$), the determinant E_α is $\neq 0$ for every positive integer α and the coefficients $F_\nu(\eta_\chi)$ are defined unambiguously by (19) for $\nu = 1, 2, \dots, n$ and for every system $(\eta_1, \dots, \eta_t)$ of degree α. In this manner we have proved:

If $\Omega = 0$, the considered functional system possesses at most one solution analytical and vanishing at the origin $x_\tau = 0$.

Further we find:

If Ω is positive, the functional system possesses at most one solution $(f_\nu(x_\tau))$, where $f_\nu(x_\tau)$ is analytical at the origin $x_\tau = 0$, takes at that point the value zero and is in the vicinity of the origin $x_\tau = 0$ approximatively equal to a given polynomial $p_\nu(x_\tau)$ of degree Ω.

In fact, the coefficients $F_\nu(\eta_\chi)$ are defined unambiguously by (19) for every system $(\eta_1, \dots, \eta_t)$ of degree $\alpha > \Omega$.

The theorem is therefore establisbed as soon as we have shown that the power series found for $f_\nu(x_\tau)$ possesses in any case a positive radius of convergence. Indeed these series give a system of n functions $f_\nu(x_\tau)$ which possesses all required properties. It is to observe that in that case also the last assertion has been proved. In fact, suppose there exists a positive integer α such that $D_\alpha = 0$. Let Ω be the greatest integer with this property. I may suppose that there exists at least one system of n polynomials $p_\nu^*(x_\tau)$ of degree Ω such that $p_\nu^*(x_\tau) = 0$ at $x_\tau = 0$ and that each function $g_\varrho\{x_\tau, p_\nu^*(l_{\chi\omega}(x_\tau))\}$ possesses at the origin $x_\tau = 0$ a zero with multiplicity $> \Omega$, for else the functional system does not possess any solution analytical and vanishing at the origin $x_\tau = 0$. If $p_\nu(x_\tau)$ denotes an arbitrary polynomial of degree Ω containing the same terms of degree $< \Omega$ as $p_\nu^*(x_\tau)$, then $g_\varrho\{x_\tau, p_\nu(l_{\chi\omega}(x_\tau))\}$ possesses at $x_\tau = 0$ a zero with multiplicity $> \Omega - 1$. The above argument, applied on $f_\nu(x_\tau)$, $y_{k\varrho} - h_\varrho(x_\tau, y_{\mu\nu})$ and $S_{\nu\varrho}(\eta_\chi, \zeta_\varphi)$, can be applied on $p_\nu(x_\tau)$, $g_\varrho(x_\tau, y_{\chi\nu})$ and $T_{\nu\varrho}(\eta_\chi, \zeta_\varphi)$. In this manner we find that the coefficient of $x_1^{\zeta_1} \dots x_t^{\zeta_t}$ in $g_\varrho \{ x_\tau, p_\nu (l_{\chi\omega}(x_\tau) \}$ is for any system $(\zeta_1, \dots, \zeta_t)$ of degree Ω equal to

$$\sum_{\nu, \eta_\chi} P_\nu(\eta_\chi)\, T_{\nu\varrho}(\eta_\chi, \zeta_\varphi) - u_\varrho^*(\zeta_\varphi) \qquad (\Sigma\, \eta_\chi = \Omega),$$

where $P_\nu(\eta_\chi)$ is the coefficient of $x_1^{\eta_1} \dots x_t^{\eta_t}$ in $p_\nu(x_\tau)$ and $u_\varrho^*(\zeta_\varphi)$ is defined unambiguously since in $p_\nu(x_\tau)$ the coefficients of the terms of degree $< \Omega$ are already known. Hence the n functions $g_\varrho \{ x_\tau, p_\nu (l_{\chi\omega}(x_\tau)) \}$ possess at $x_\tau = 0$ a zero with multiplicety $> \Omega$ if and only if the relations

$$\sum_{\nu, \eta_\chi} P_\nu(\eta_\chi)\, T_{\nu\varrho}(\eta_\chi, \zeta_\varphi) = u_\varrho^*(\zeta_\varphi) \qquad (\Sigma\, \eta_\chi = \Omega)$$

are satisfied for $\varrho = 1, 2, \dots, n$ and for every system $(\zeta_1, \dots, \zeta_t)$ of degree Ω. In this manner we find $\binom{\Omega + t - 1}{t - 1} n$ linear equations with as many unknown numbers $P_\nu(\eta_\chi)$. This system possesses at least one solution, since it is satisfied if we take for $P_\nu(\eta_\chi)$ the coefficient of $x_1^{\eta_1} \dots x_t^{\eta_t}$ in $\overset{*}{p_\nu}(x_\tau)$. The determinant D_Ω being zero, the system of equations possesses an infinity of solutions. The number of systems $(p_\nu(x_\omega))$ for which the n functions $g_\varrho \{x_\tau, p_\nu (l_{\chi\omega}(x_\tau))\}$ have at $x_\tau = 0$ a zero with multiplicity $> \Omega$ is therefore infinite. Each of these systems $(p_\nu(x_\tau))$ gives a solution $(f_\nu(x_\tau))$ of the functional system with the property that $f_\nu(x_\tau)$ is in the vicinity of $x_\tau = 0$ analytical and approximatively equal to $p_\nu(x_\tau)$. Hence the functional system possesses an infinity of solutions analytical and vanishing at the origin $x_\tau = 0$.

III. Proof that the radii of convergence are positive.

Since $E_\alpha \neq 0$ for $\alpha > \Omega$ and $E_\alpha \to 1$ for $\alpha \to \infty$, $| E_\alpha |$ possesses for $\alpha > \Omega$ a positive lower bound $\frac{1}{A}$ independent of α. From (19) it follows that

$$| F_\nu(\eta_\chi) | \leqq A \, \Big| \sum_{\varrho, \zeta_\varphi} S_{\nu,\varrho}^*(\eta_\chi, \zeta_\varphi)\, u_\varrho(\zeta_\varphi) \Big| \qquad (\sum_\varphi \zeta_\varphi = \alpha)$$

for $\nu = 1, 2, \dots, n$ and for every system $(\eta_1, \dots, \eta_t)$ of degree $\alpha > \Omega$; in this formula $S_{\nu,\varrho}^*(\eta_\chi, \zeta_\varphi)$ denotes the minor of $S_{\nu,\varrho}(\eta_\chi, \zeta_\varphi)$ in E_α. If we deal with $S_{\nu,\varrho}^*(\eta_\chi, \zeta_\varphi)$ in the same manner as with E_α, formule (24) becomes

$$| S_{\nu,\varrho}^*(\eta_\chi, \zeta_\varphi) - 1 | < 2\, C \binom{\alpha + t - 1}{t - 1} n\, \theta^\alpha \text{ for } \nu = \varrho, \eta_\chi = \zeta_\chi,$$

$$| S_{\nu,\varrho}^*(\eta_\chi, \zeta_\varphi) | < 2\, C \binom{\alpha + t - 1}{t - 1} n\, \theta^\alpha \quad \text{in the other cases,}$$

if $\alpha > \Omega$ is so large that $C \binom{\alpha + t - 1}{t - 1} n\, \theta^\alpha < \frac{1}{2}$. Since $\sum\limits_{\varrho, \zeta\varphi}$ is a sum of $\binom{\alpha + t - 1}{t - 1} n$ terms, we obtain for sufficiently large α

$$| F_\nu(\eta_\chi) | \leqq A \left\{ 1 + 2\, C \binom{\alpha + t - 1}{t - 1}^2 n^2\, \theta^\alpha \right\} Max\, | u_\varrho(\zeta_\varphi) | \qquad (\sum_\varphi \zeta_\varphi = \alpha),$$

hence for $\alpha > \Omega$ $\quad | F_\nu(\eta_\chi) | \leqq B\, Max\, | u_\varrho(\zeta_\varphi) | \qquad (\sum\limits_\varphi \zeta_\varphi = \alpha)$, . . . (26)

where B is independent of $\nu, \eta_1, \eta_2, \dots, \eta_t$ and where $Max\, | u_\varrho(\zeta_\varphi) |$ is the greatest of the $\binom{\alpha + t - 1}{t - 1} n$ numbers $| u_\varrho(\zeta_\varphi) |$ $(1 \leqq \varrho \leqq n, \sum\limits_\varphi \zeta_\varphi = \alpha)$.

The functions $q_\nu(x_\tau)$ and $w_{\mu\chi}(x_\tau)$ are analytical at $x_\tau = 0$; the functions $h_\varrho(x_\tau, y_{\mu\nu})$ are analytical at $x_\tau = y_{\mu\nu} = 0$. Hence there exist functions $\overline{q}_\nu(x_\tau)$ and $\overline{w}_{\mu\chi}(x_\tau)$ that are analytical at $x_\tau = 0$ and further n functions $\overline{h}_\varrho(x_\tau, y_{\mu\nu})$ that are analytical at $x_\tau = y_{\mu\nu} = 0$ with the properties

$$q_\nu(x_\tau) << \overline{q}_\nu(x_\tau); \;\; w_{\mu\chi}(x_\tau) << \overline{w}_{\mu\chi}(x_\tau); \;\; h_\varrho(x_\tau, y_{\mu\nu}) << \overline{h}_\varrho(x_\tau, y_{\mu\nu})$$

and
$$\left|\left(\frac{\partial w_{\mu\chi}}{\partial z_\omega}\right)_0\right| = \left(\frac{\partial \overline{w}_{\mu\chi}}{\partial z_\omega}\right)_0,$$
so that it follows from (23) that
$$\sum_\omega \left(\frac{\partial \overline{w}_{\mu\chi}}{\partial z_\omega}\right) \leqq \theta < \tfrac{1}{2}(1+\theta). \qquad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If the positive number Γ is small enough, $\overline{h}_\varrho(x_\tau, y_{\mu\nu})$ is analytical for $|x_\tau| \leqq \Gamma$, $|y_{\mu\nu}| \leqq \Gamma$. If the positive number $K \leqq \Gamma$ is small enough, then $\overline{q}_\varphi(z_\tau)$ and $\overline{w}_{\mu\chi}(z_\tau)$ are analytical for $|z_\tau| \leqq K$ and we have for $|z_\tau| \leqq K$
$$|\overline{q}_\varphi(z_\tau)| \leqq \Gamma \text{ and } |\overline{w}_{\mu\chi}(z_\tau)| < \tfrac{1}{2}(1+\theta)K \qquad . \quad . \quad . \quad . \quad (28)$$
by (27). Put
$$b_\nu(x_\chi) = \Lambda\left\{-1 + \prod_\chi\left(1 - \frac{2x_\chi}{(1+\theta)K}\right)^{-1}\right\}.$$

If $|z_\tau| \leqq K$, then (28) holds and $b_\nu(\overline{w}_{\mu\chi}(z_\tau))$ is therefore analytical. I choose the positive number Λ so small that
$$|b_\nu(\overline{w}_{\mu\chi})(z_t)| \leqq \Gamma \text{ for } |z_\tau| \leqq K.$$

Hence the function $\overline{h}_\varrho\{\overline{q}_\varphi(z_\tau), b_\nu(\overline{w}_{\mu\chi}(z_\tau))\}$ is analytical for $|z_\tau| \leqq K$ and therefore $<< M \prod_\tau \left(1 - \frac{z_\tau}{K}\right)^{-1}$, if M is large enough. I choose the positive integer $N \geqq \Omega$ so large that
$$BM\left(\frac{1+\theta}{2}\right)^{N+1} < \Lambda \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$
and further $H \geqq \frac{2}{(1+\theta)K}$ so large that the inequalities
$$|F_\nu(\zeta_\varphi)| \leqq \Lambda H^{\zeta_1 + \ldots + \zeta_t} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$
hold for $\nu = 1, 2, \ldots, n$ and for any system $(\zeta_1, \ldots, \zeta_t)$ of positive degree $\leqq N$. It is sufficient to prove that the inequalities (30) hold for $\nu = 1, 2, \ldots, n$ and for any system $(\zeta_1, \ldots, \zeta_t)$ of positive degree. In fact, then the power series obtained for $f_\nu(x_\tau)$ converges for $|x_\tau| < \frac{1}{H}$.

I may assume that $(\zeta_1, \ldots, \zeta_t)$ has a degree $\alpha \geqq N+1$ and that (30) has already been proved, if $(\zeta_1, \ldots, \zeta_t)$ is replaced by a system of positive degree $< \alpha$. We find for the function $j_\nu(x_\tau)$ defined by (20)
$$j_\nu(x_\tau) << \Lambda\{-1 + \prod_\tau (1 - Hx_\tau)^{-1}\} = b_\nu\{\tfrac{1}{2}(1+\theta)KHx_\tau\}.$$

Hence it follows from $\frac{1}{2}(1+\theta)KH \geqq 1$ that
$$j_\nu(w_{\mu\chi}(z_\tau)) << b_\nu\{\tfrac{1}{2}(1+\theta)KH\overline{w}_{\mu\chi}(z_\tau)\} << b_\nu(\overline{w}_{\mu\chi}(v_\tau)),$$
where $v_\tau = \frac{1}{2}(1+\theta)KH z_\tau$, and that
$$\overline{q}_\varphi(z_\tau) << \overline{q}_\varphi(v_\tau).$$

In this manner we obtain
$$h_\varrho\{q_\varphi(z_\tau), j_\nu(w_{\mu\chi}(z_\tau))\} << \overline{h}_\varrho\{\overline{q}_\varphi(v_\tau), b_\nu(\overline{w}_{\mu\chi}(v_\tau))\} << M\prod_\tau\left(1 - \frac{v_\tau}{K}\right)^{-1} = M\prod_\tau\{1 - \tfrac{1}{2}(1+\theta)Hz_\tau\}^{-1}$$

The coefficient of $z_1^{\zeta_1} \ldots z_t^{\zeta_t}$ in the expansion of the left-hand side is $u_\varrho(\zeta_\varphi)$, so that the absolute value of this number is
$$\leqq M(\tfrac{1}{2}(1+\theta)H)^\alpha < \frac{\Lambda}{B}H^\alpha$$
by (29) and $\alpha \geqq N+1$. Hence (30) follows from (26), which proves the theorem.

Physiology. — *Body temperature as a variable factor in the energy balance of the organism.* By G. VAN RIJNBERK.

(Communicated at the meeting of January 31, 1942.)

The significance of the glycogen in the liver, and of the subcutaneous fat, for the material, and energetic, balances of the organism is evident: both are to be considered as depots where material, and energetic, reserves are stored. When composing the balance sheet of the material ingesta and egesta it is necessary to consider the possibility of changes in body weight, an increase, or decrease, of which can explain apparent deviations from metabolic equilibrium. When balancing the energetic income against the output of energy, it is also necessary to take into account any changes in body weight which may have occurred in the course of the experiment. A shortage of energetic output may depend on the fact that part of the chemical energy obtained from the food has been retained within the body as storage material. Conversely, an apparent surplus of energetic egesta may be occasioned by the fact that this storage material has been called upon to furnish energy. This elementary fact is known to any investigator composing such balance sheets.

As far as I know, less attention seems to have been given to body temperature as a variable factor in the energy balance, despite the fact that an appreciable movement of heat is necessary for any change in body temperature.

Assuming the specific heat of the human body as a whole to equal 0.8, a change in body temperature of one degree Centigrade will, if the body weight be 70 kilograms, correspond to

$$1 \times 70 \times 0.8 \text{ Calories} = 56 \text{ Cal.}$$

In the following, the theoretical significance of this fact will be illustrated by means of a few examples.

1. Increase of body mass.

In the course of growth the body weight of an individual will increase from, say, 6 kilograms, when a young child, to 70 kilograms. If all the food taken was at room temperature (20 degrees Cent.), then the heat needed to bring the additional 64 kg to body temperature (37 degrees Cent.) will have been equal to

$$17 \times 64 \times 0.8 \text{ Cal.} = 870.4 \text{ Cal.}$$

2. Increase of body temperature through muscular activity.

The temperature of the body shows a more or less marked increase in all cases where mechanical work is performed. Let us take a simple case of the kind usually found in textbooks.

An amount of work equal to 222×10^3 kilogrammeters, performed in two hours, led to a rise of body temperature amounting to 0.72 degrees Cent. Assuming the body weight to have been 70 kg, this means that the heat produced by the muscular work has in part been used to increase the temperature of the body, the amount of heat in question being equal to

$$0.72 \times 70 \times 0.80 \text{ Cal.} = 40 \text{ Cal.}$$

Now 40 Calories are equivalent to 17080 kgm, or one thirteenth (8 per cent.) of the total work done. Of course, these 40 Cal. do not remain in the body, the rise in body temperature being transient. When investigating the heat output of a subject performing mechanical work by means of a calorimeter, these 40 calories will eventually be found back in the calorimeter, as soon as the body temperature has again fallen to its original level; consequently, they will, in time, appear among the egesta. If the duration of the experiment should be too short, however, the heat output would be less than the amount of heat really generated.

3. Rise of body temperature after heat puncture.

Assuming the average body temperature of the rabbit to be 39 degrees Cent., its body weight 3 kg, and the temperature attained in the experiment 42 degrees Cent., then the amount of heat which was needed to produce this rise in body temperature equals

$$3 \times 3 \times 0.8 \text{ Cal.} = 7.2 \text{ Cal.}$$

The quantity of heat used for this additional rise in body temperature is relitively very large, since it amounts to 2.4 Cal. per kg body weight, or hardly less than the normal total heat production of a rabbit, per hour and kg. In those cases where the temperature reaches its maximum in two hours, reckoned from the time at which it starts to rise, the heat used to increase the body temperature alone will necessitate an increase of the heat production of nearly 50 per cent. In those cases where the maximum temperature is reached in one hour, the normal heat production will have to be nearly doubled to account for the rise in temperature alone.

4. Daily fluctuations of body temperature.

In man, the daily variations of body temperature may attain an amplitude of about 1.2 degrees Cent., for an early-morning minimum of 36°3 and a maximum of 37°5 in the evening. For an individual weighing 70 kg the amount of heat involved in this is equal to

$$1.2 \times 70 \times 0.8 \text{ Cal.} = 67.2 \text{ Cal.}$$

It is a well-known fact that the intensity of metabolism, determined from the amount of carbon dioxide exhaled, runs parallel to the temperature of the body. But the output of heat, measured calorimetrically, does not keep pace with these. It seems reasonable to explain the difference between the amount of heat produced (as apparent from the production of carbon dioxide) and the amount discharged on the assumption that the difference is levelled out by the variations in body temperature.

The examples given may suffice. They show clearly that the amount of heat involved in changes of body temperature is relatively high. This heat may be considered as a kind of thermic storage material. The diagram given below may serve to make our meaning clear.

DIAGRAM.

Ingesta.	Metabolism.	Egesta.
1. material ingesta;	3. building up of living matter: anabolism; 4. breaking down of living matter: catabolism;	7. material egesta; 8. energetic egesta: a. contained in excreta;
2. energetic ingesta.	5. rise of energy level: ectropy; 6. fall of energy level: entropy.	b. heat (radiation, &c.); c. mechanical work.

Positive or negative storage:

(material) 9. increase } of body 10. decrease } weight	11. increase } of energy 12. decrease } store	(energetic) a. bound chemical energy; b. caloric energy (body temperature).

Legenda.

1. material intake, considered from a strictly material point of view;
2. material intake, considered as a source of energy only;
3. increase of the amount of living matter through assimilation of food;
4. decrease of the amount of living matter through dissimilation;
5. increase of the energy level of the organism through storage of bound energy;
6. decrease of the energy level of the organism through liberation of hitherto bound energy. This may occur:
   a. through generation of heat, without motion or secretion (muscle tone, functional activity of central nervous system, &c.);
   b. through generation of heat, accompanied by motion or secretion;
7. material losses (excreta, carbon dioxide, water), considered from a strictly material point of view;
8. a. material losses, considering the small amount of bound energy they still possess;
   b. heat discharged (conduction, convection, evaporation);
   c. energy discharged as mechanical work;

9, 10. gain or loss of body weight;

11, 12. rise or fall of the energy level of the organism, viz.
   a. through modification of the amount of bound (chemical) energy, running parallel to 9. and 10.;
   b. through fluctuations of the body temperature, considered as a measure of the caloric stores of the organism.

Fall of body temperature means a lowering of the energy level of the body, considered as a thermo-energetic system; rise of body temperature means an increase of this level.

One restriction should be made. The retaining by the body of a given amount of heat, leading to a rise of body temperature, has in the foregoing been considered as a positive storage. The fact should be stressed, however, that, in the case considered under 11b this kind of storage is far less important for the economy of the body than the storage of chemically bound energy. Fat and glycogen are stable and lasting energy stores, which can be kept indefinitely, and which can be drawn upon at any time. The increase of the energy level expressing itself as a temporary rise in body temperature, on the other hand, is very unstable; the surplus heat tends to flow away, as it should, the body striving to regain the temperature level characteristic for the species in question.

The possibility remains, however, that the extra store of heat is not quite lost to the body, inasmuch as, during the flowing away of the surplus heat, the generation of heat can be lessened to some extent at least.

Medicine. — *Spreading of gliadin.* I. By E. GORTER and P. C. BLOKKER.

(Communicated at the meeting of February 28, 1942.)

Introduction.

In the investigations on the spreading of various proteins made by one of us in collaboration with GRENDEL [1]), a few experiments were also made with gliadin. Of late years several experimenters have investigated the spreading of this remarkable protein more thoroughly [2]), [3]), [4]). Moreover in the last mentioned investigation COCKBAIN and SCHULMAN studied the influence of tannic acid on gliadin films. As the results of the experiments with gliadin show a marked difference with those carried out with most of the other proteins, it seemed to us worthwhile to make a closer study of the properties of gliadin films. Particularly we investigated the influence of the pH on the spreading pressure, the surface potential and the surface viscosity.

Apparatus:

The method of *spreading* invariably used by us consists in carefully blowing a small quantity of protein solution on to the water surface by means of a micropipette.

Experiments.

The gliadin was prepared from wheat gluten according to the method described by DILL and ALSBERG [5]). The gliadin was obtained in a very pure state. The nitrogen content of the dried substance was 17.3 %. The liquid spread was a clear solution of gliadin in 60 per cent (by volume) aqueous ethyl alcohol. The concentration of the solution was invariably 0.58 %, in the viscosity experiments only the concentration was twice as great.

To study the influence of the pH on the properties of gliadin films the spreading was effected on aqueous solutions of different pH. The solutions used for pH 3.5—5.6 were 1/300 mol. sodium acetate-acetic acid buffers. The solutions of higher acidity were made with hydrochloric acid. For pH 5.6—9.0 1/300 mol. veronal-acetate buffer solutions as described by MICHAELIS [6]) were used. The solutions of higher alkalinity were made with sodium carbonate. The pH of the solutions was controlled by measurements with a hydrogen electrode; a sensitive mirror galvanometer served as zero instrument.

In connection with the publication of COCKBAIN and SCHULMAN [4]) the influence of the pH on the properties of gliadin-tannic acid films was also studied. The tannic acid (KAHLBAUM pro analyse) was purified by extraction with petroleum spirit in order to remove traces of fats etc. which might be present. However, the preparation still contained impurities after this treatment. This was shown in blank experiments with solutions of tannic acid in the buffers (about 100 mg. per l. buffer); notwithstanding repeated cleaning of the surface of the solution with a paraffined glass rod, after some time rather high pressures were found when the surface was compressed.

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[1]) E. GORTER and F. GRENDEL, Proc. Acad. Sc. A'dam **29**, 1262 (1926); Trans. Faraday Soc. 22, 477 (1926).

[2]) J. S. MITCHELL, Trans. Faraday Soc. **33**, 1129 (1937).

[3]) A. H. HUGHES and E. K. RIDEAL, Proc. Roy. Soc. A **137**, 62 (1932).

[4]) E. G. COCKBAIN and J. H. SCHULMAN, Trans. Faraday Soc. **35**, 1266 (1939).

[5]) D. B. DILL and C. L. ALSBERG, J. biol. Chem. **65**, 279 (1925).

[6]) L. MICHAELIS, Bioch. Z. **234**, 139 (1931).

Pressure-area measurements on gliadin and gliadin-tannic acid films.

In fig. 1 are shown some pressure-area curves of gliadin at different pH values. These curves do not contain a linear part as is the case with most proteins. They satisfy the equation of a rectangular hyperbole, which indicates that the films are of the liquid type.

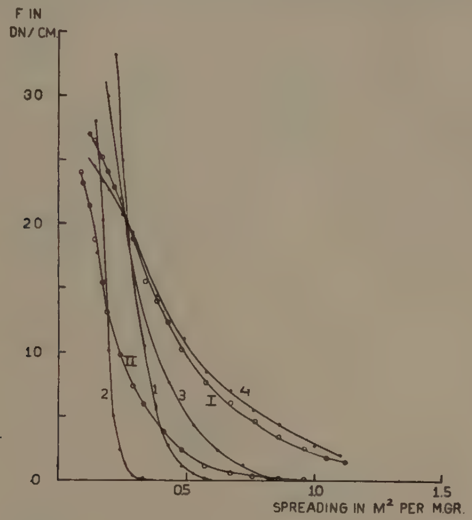

Fig. 1. Surface pressure as a function of the area occupied by one mg. substance for gliadin and gliadin-tannic acid films.

I = curve for gliadin film at pH = 2.0
II = „ „ „ „ „ pH = 3.6
1 = curve for gliadin-tannic acid film at pH = 2.0 and 5.9
2 = „ „ „ „ „ „ „ pH = 2.9
3 = „ „ „ „ „ „ „ pH = 7.5
4 = „ „ „ „ „ „ „ pH = 11.1

For such expanded films LANGMUIR [7]) derived the equation:

$$(F-F_s)\,(a-a_0) = kT$$

where F = pressure of the film, a = area of spread, T = abs. temperature and F_s, a_0 and k are constants; a_0 is analogous to the b-term in the VAN DER WAALS equation. However, with gliadin in most cases negative a_0 values were found and only in a few cases very small positive a_0 values, so that the equation looses much of its physical meaning in the case of gliadin. For that reason we have refrained from characterizing our curves by this equation. The negative a_0 values may be explained as follows: the measurements showed that in the case of rather high pressures after compressing the film the pressure decreased slowly (sometimes 10 %) and after some time became constant. These final pressures could be reproduced very well. Probably this phenomenon is caused by squeezing out hydration water from between the micelles of the film. On diminishing the pressure the reverse phenomenon was observed.

On account of the lack of a linear part in the pressure-area curves, it is not possible to extrapolate the curves in the usual way to zero-force. COCKBAIN and SCHULMAN's curves do not contain a linear part either; their curves, however, are less bent than ours. In a former publication [8]) SCHULMAN and RIDEAL plotted the pressure against the surface concentration. The curves obtained contain a considerable linear part. In extrapolating these curves to zero-force one finds a concentration of about 0.8 and 0.6 × 10^{-7} g cm^2 at pH respectively 2.0 and 5.9 which corresponds to an area of about 1.3 resp. 1.6 m^2/mg. In extrapolating our curves in this way we find an area of 1.5 resp. 1.3 m^2/mg. Finally, it may be stated that GORTER and GRENDEL 1) neither found a linear part in the pressure-area curves at room temperature, but found one at 40° C. We could not reproduce this latter observation; at 40° C. we found the same type of curves as at room temperature.

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[7]) I. LANGMUIR, J. chem. Physics **1**, 756 (1933).

[8]) J. H. SCHULMAN and E. K. RIDEAL, Proc. Roy. Soc. B **122**, 29 and 46 (1937).

The variations in the curves found by different authors may be explained by variations in the manner of preparation of the gliadin, which may influence the character of the protein and therefore also the pressure-area relations obtained.

As extrapolating the pressure-concentration curves to zero pressure in our case has little physical meaning and, moreover, confusion may occur with values for other proteins, obtained in the usual way of extrapolating pressure-area curves, we dropped the first mentioned extrapolation; therefore we only give the area for some arbitrary pressures (see fig. 2).

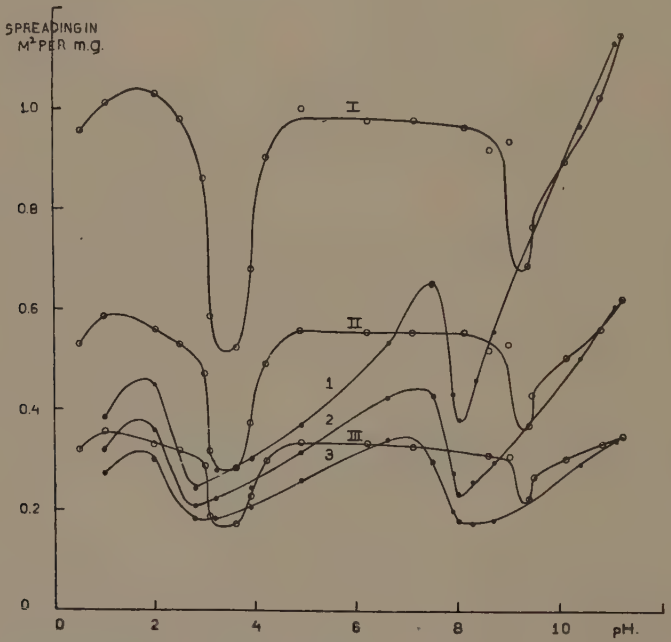

Fig. 2. Spreading of gliadin and gliadin-tannic acid films as a function of the pH.
I, II and III curves for gliadin at surface pressures of resp. 2, 8 and 16 dn/cm.
1, 2 and 3 curves for gliadin-tannic acid at the same surface pressures.

The area of spread for gliadin at different pH values for three arbitrary pressures is shown in fig. 2. The maximum spreading occurs in the vicinity of the isoelectric point (pH about 6.5). Contrary to other proteins this maximum is not a sharp one; moreover in the minima the spreading is rather great. Possibly both phenomena are due to the solvent, as alcohols often increase the spreading of proteins 9). With respect to the influence of electrolytes on the area of spread gliadin behaves as the other proteins. In consequence of the high concentration of H^+ or OH^- ions the area occupied by

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9) L. FOURT and A. PERLEY, Proc. Soc. exptl. Biol. Med. **33**, 201 (1935); S. STÄLLBERG and T. TEORELL, Trans. Faraday Soc. **35**, 1413 II (1939).

gliadin at very low or high pH values is great; addition of 0.01 gr. eq. SO_4^{--} to buffer solutions that give minimal spreading at the acid side or 0.01 gr. eq. Ca^{++} to those on the alkaline side, strongly enlarges the area of spread.

The effect of tannic acid on gliadin films was studied by spreading the gliadin solution on buffers containing 100 mg tannic acid per l.. Higher concentrations of tannic acid only slightly increased the effect found. On the acid side the type of curves has altered entirely; the steep linear part (see fig. 1) indicates that the films are solid. COCKBAIN and SCHULMAN who injected tannic acid underneath gliadin films already present 4) found that the films obtained collapsed at pressures greater than about 0.5 dn/cm. In our experiments we stated a similar phenomenon. We doubt, however, if this is to be considered as collapsing; after each compression the pressure rises very much and then decreases considerably, but after some time the pressure becomes practically constant at a pressure yet much higher than before. So the curves shown in fig. 1 are obtained by waiting after each compression till the pressure was practically constant.

The reproducibility of the area at which the first measurable pressure occurred was rather bad. This is not to be wondered at as the area occupied by the protein will be mainly determined by the velocity of spreading of the gliadin solution and by the velocity of reaction between gliadin and tannic acid. The gliadin-tannic acid complex itself only spreads slowly or not at all; this was shown by the fact that a compressed gliadin-tannic acid film, even after waiting a long time, exerted no or only a small force on the differential balance when the surface was enlarged. It is even possible that the complex films obtained with our method were not monomolecular in all places and had no uniform thickness. Therefore one should not attach much importance to the areas obtained by extrapolating the curves to zero-pressure.

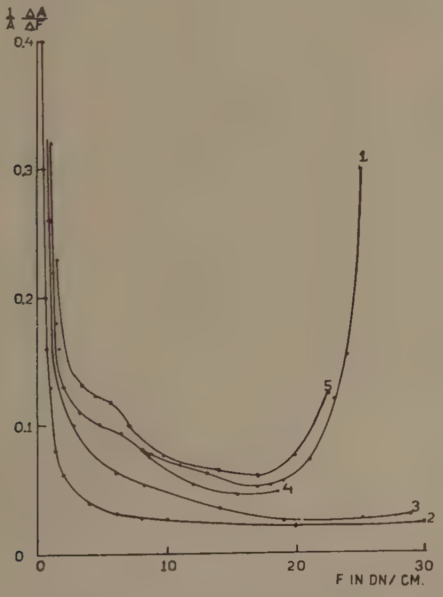

Fig. 3. Compressibility of gliadin films and of gliadin-tannic acid films at different surface pressures *F*.

1 = gliadin film for pH = 1—11
2 = gliadin-tannic acid film for pH = 1—6.6
3 = ,, ,, ,, ,, ,, pH = 7.5—7.9
4 = ,, ,, ,, ,, ,, pH = 8.7
5 = ,, ,, ,, ,, ,, pH = 10.4

Fig. 1 and 2, and also all further observations with potential- and viscosity measurements, show that above pH 6—7 the complex films behave more and more as gliadin films when the pH increases. At pH = 11 the influence of tannic acid has disappeared.

An interesting result is obtained when we calculate the *compressibility* $-\frac{1}{A}\frac{dA}{dF}$ (the reverse of surface elasticity). Fig. 3 shows that the compressibility of gliadin is equal at all pH values. Therefore it is very likely that the pH has no influence on the character of the film, but only affects the amount of gliadin that collects in the surface.

Above a pressure of about 20 dn/cm the compressibility strongly increases. Evidently the molecules are pressed out of the film at high pressures and are pushed over each other.

The curves for gliadin show a small bend at a pressure of about 6 dn/cm. This is probably connected with initial gelation (see f.i. 4) at this pressure, a phenomenon that will be dealt with in more detail in treating the viscosity measurements. As the deviation is of the same order as the error made in deriving the compressibility (± 0.01) it is not to be excluded, however, that the bend has to be ascribed to accidental causes.

Finally the compressibility curves show clearly that the tannic acid diminishes the compressibility of gliadin films considerably at pH values below 7 and that this influence strongly decreases at higher pH.

(To be continued.)

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Applied Mechanics. — *On the state of stress in perforated strips and plates.* By K. J. SCHULZ. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of February 28, 1942.)

1. *Introduction.* In a former treatise (which in this paper will be indicated by the letter "A") the stress-problem of infinite plates containing circular holes of arbitrary position and arbitrary radii has been studied at great length. Special attention was drawn to such problems in which equal holes were arranged in one or more infinite rows of constant pitch and in which the states of stress and strain showed periodicity with a period equal to that pitch. Further investigation of these problems led to the result that by renewed use of this periodicity a whole class of other problems can be made accessible by the same method, namely those problems which bear upon the semi-infinite plate containing one or more rows of equal holes parallel to the edge of the plate, respectively upon a strip of infinite width with one or more rows of holes parallel to its edges. Though the holes of each row must be of equal radius, it is not necessary that this equality holds for the holes of different rows. On the other hand it is required, that the geometrical configuration of the holes and the loadsystem of plate or strip possess a common period. As to the strip, the latter condition is fulfilled in the cases of pure tension and pure bending; bending accompanied with shear requires a separate treatment.

2. *The method of investigation.* The starting point of our investigation is the well known stress function F of Airy from which the stress-components in Cartesian coordinates y, z (conceived as the average over the thickness of the plate, which furthermore will be put equal to unity) are to be derived by

$$\sigma_y = \frac{\partial^2 F}{\partial z^2}, \qquad \sigma_z = \frac{\partial^2 F}{\partial y^2}, \qquad \tau_{yz} = -\frac{\partial^2 F}{\partial y\,\partial z}. \quad . \quad . \quad . \quad (1)$$

The function itself is defined by the bi-harmonic equation

$$\Delta' \Delta' F = 0 \qquad \left(\Delta' = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right); \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the stress-conditions which relate to the inner and outer boundaries of the plate.

In polar coordinates, r, φ

$$y = r \cos \varphi, \qquad z = r \sin \varphi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the stress-components are expressed by

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{r}\frac{\partial F}{\partial r}, \qquad \sigma_\varphi = \frac{\partial^2 F}{\partial r^2}, \qquad \tau_{r\varphi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \varphi}\right), \quad . \quad (4)$$

whereas the bi-harmonic equation takes the form

$$\Delta' \Delta' F = 0 \qquad \left(\Delta' = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right). \quad . \quad . \quad . \quad (5)$$

The relations between the sets of stresses $\sigma_y, \sigma_z, \tau_{yz}$, and $\sigma_r, \sigma_\varphi, \tau_{r\varphi}$ are given by

$$\left.\begin{aligned} \sigma_r &= \tfrac{1}{2}(\sigma_y + \sigma_z) + \tfrac{1}{2}(\sigma_y - \sigma_z)\cos 2\varphi + \tau_{yz}\sin 2\varphi, \\ \sigma_\varphi &= \tfrac{1}{2}(\sigma_y + \sigma_z) - \tfrac{1}{2}(\sigma_y - \sigma_z)\cos 2\varphi - \tau_{yz}\sin 2\varphi, \\ \tau_{r\varphi} &= \qquad\qquad\quad - \tfrac{1}{2}(\sigma_y - \sigma_z)\sin 2\varphi + \tau_{yz}\cos 2\varphi. \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

Finally, if we have to deal with a multi-connected plate, certain conditions of unambiguity have to be regarded, to which we will refer later on.

With the aid of the just-mentioned general formulae two particular systems of stress-functions are constructed, one of which relates to the infinite plate containing one single circular hole; the other one bears upon the semi-infinite plate. These systems admit the excitement of prescribed stresses along the edge of the circular hole, resp. the excitement of prescribed (periodical) stresses along the straight boundary of the semi-infinite plate, and both of them give rise to stresses which at infinity tend to zero. These two systems are the subject of paragraphs 3 and 5. In par. 4, which relates to the first system of stress-functions, formulae are derived for the stress components in points of a straight line parallel to the y-axis, resp. the axis of the polar system of coordinates. Par. 6 relates to a similar question, now with respect to the half infinite plate: formulae are given for the stresses in points of a circular boundary provided that a given set of stresses acts upon the straight boundary of the plate. The results obtained in these paragraphs find their application in §§ 7 and 8, where the symmetrical strip with one single row of holes is treated, firstly subjected to pure tension, secondly to pure bending. Par. 9 treats the problem of the semi-infinite plate, with one row of holes (parallel to its edge), loaded by (equal) resultant forces.

3. *The stress-functions relating to one single row of holes.* If one circular hole (radius a) in an infinite plate is loaded by the equilibrium-system of boundary-stresses

$$\left.\begin{aligned} \sigma_r &= C_0 + \sum_{n=1}^{\infty} (C_n \cos n\varphi + \overline{C}_n \sin n\varphi), \\ \tau_{r\varphi} &= \sum_{n=1}^{\infty} (D_n \sin n\varphi + \overline{D}_n \cos n\varphi) \end{aligned}\right\} \text{ with } C_1 = D_1,\ \overline{C}_1 = -D_1 \qquad (1)$$

the state of stress in the plate itself is governed by the stress-function

$$F = A_0 F_0 + A_1 F_1 + \overline{A}_1 \overline{F}_1 + \sum_{n=1}^{\infty} (A_n F_n + B_n F_n^* + \overline{A}_n \overline{F}_n + \overline{B}_n \overline{F}_n^*), \qquad (2)$$

(comp. A 3, 13), with (see A 3, 8)

$$\left.\begin{aligned} &A_0 = C_0 a^2,\ A_1 = -\tfrac{1}{2} C_1 a^3,\ \overline{A}_1 = -\tfrac{1}{2} \overline{C}_1 a^3, \\ &A_n = \frac{n C_n - (n+2) D_n}{2n(n+1)} a^{n+2},\quad B_n = -\frac{C_n - D_n}{2(n-1)} a^n, \\ &\overline{A}_n = \frac{n \overline{C}_n + (n+2) \overline{D}_n}{2n(n+1)} a^{n-2},\quad \overline{B}_n = -\frac{\overline{C}_n + \overline{D}_n}{2(n-1)} a^n \quad (n \geqq 2) \end{aligned}\right\} \qquad (3)$$

and (see A 3, 14).

$$\left.\begin{aligned} &F_0 \equiv \ln r,\ F_1 \equiv r^{-1} \cos\varphi,\ \overline{F}_1 \equiv r^{-1} \sin\varphi, \\ &F_n \equiv r^{-n} \cos n\varphi,\ F_n^* \equiv r^{-n+2} \cos n\varphi,\ \overline{F}_n \equiv r^{-n} \sin n\varphi,\ \overline{F}_n^* \equiv r^{-n+2} \sin n\varphi (n \geqq 2). \end{aligned}\right\} \qquad (4)$$

If the constants (3) are substituted into (2), the function F may be written as

$$F = C_0 F_{\sigma 0} + \sum_{n=1}^{\infty} (C_n F_{\sigma n} + D_n F_{\tau n} + \overline{C}_n \overline{F}_{\sigma n} + \overline{D}_n \overline{F}_{\tau n}), \quad . \quad (5)$$

where the symbols $F_{\sigma n}, F_{\tau n}, \overline{F}_{\sigma n}, \overline{F}_{\tau n}$ are determined by (see A, 3, 10)

$$\left.\begin{aligned} &F_{\sigma 0} \equiv a^2 \ln r, \quad F_{\sigma 1} + F_{\tau 1} \equiv -\tfrac{1}{2}\frac{a^3}{r}\cos\varphi, \quad \overline{F}_{\sigma 1} - \overline{F}_{\tau 1} \equiv -\tfrac{1}{2}\frac{a^3}{r}\sin\varphi, \\ &F_{\sigma n} \equiv -\tfrac{1}{2}a^2\left(\frac{1}{n-1} - \frac{1}{n+1}\frac{a^2}{r^2}\right)\left(\frac{a}{r}\right)^{n-2}\cos n\varphi, \\ &\overline{F}_{\sigma n} \equiv -\tfrac{1}{2}a^2\left(\frac{1}{n-1} - \frac{1}{n+1}\frac{a^2}{r^2}\right)\left(\frac{a}{r}\right)^{n-2}\sin n\varphi, \\ &F_{\tau n} \equiv +\tfrac{1}{2}a^2\left[\frac{1}{n-1} - \frac{n+2}{n(n+1)}\frac{a_2}{r^2}\right]\left(\frac{a}{r}\right)^{n-2}\cos n\varphi, \\ &\overline{F}_{\tau n} \equiv -\tfrac{1}{2}a^2\left[\frac{1}{n-1} - \frac{n+2}{n(n+1)}\frac{a^2}{r^2}\right]\left(\frac{a}{r}\right)^{n-2}\sin n\varphi \quad (n \geqq 2). \end{aligned}\right\} \quad . \quad (6)$$

Naturally the functions $F_{\sigma n}$, a.s.o., — which are linear combinations of the functions F_n, F_n^*, $\overline{F}_n$, $\overline{F}_n^*$ satisfy separately equation (2, 5). They are characterized by the following properties:

$F_{\sigma n}$ gives rise to the boundary stresses $\sigma_r = \cos n\varphi$, $\tau_{r\varphi} = 0$

$F_{\tau n}$ gives rise to the boundary stresses $\sigma_r = 0$, $\tau_{r\varphi} = \sin n\varphi$

$\overline{F}_{\sigma n}$ gives rise to the boundary stresses $\sigma_r = \sin n\varphi$, $\tau_{r\varphi} = 0$

$\overline{F}_{\tau n}$ gives rise to the boundary stresses $\sigma_r = 0$, $\tau_{r\varphi} = \cos n\varphi$.

It may be noticed that in (5) the functions $F_{\sigma 1}$, $F_{\tau 1}$, $\overline{F}_{\sigma 1}$, $\overline{F}_{\tau 1}$ only appear combined as $F_{\sigma 1} + F_{\tau 1}$ or $\overline{F}_{\sigma 1} - \overline{F}_{\tau 1}$, in consequence of the relations $C_1 = D_1$, $C_1 = -\overline{D}_1$ (comp. 1).

We now put

$$y + iz = re^{i\varphi} = x \text{ and } y - iz = re^{-i\varphi} = \overline{x}$$

in order to express the functions (4) as follows:

$$\left.\begin{aligned} &F_0 = \Re\mathrm{e} \ln x, \; F_n = \Re\mathrm{e}\, x^{-n}, \; \overline{F}_n = -\Im\mathrm{m}\, x^{-n} \; (n \geqq 1), \\ &F_n^* = \Re\mathrm{e}\, \overline{x} x^{-n+1}, \; \overline{F}_n^* = -\Im\mathrm{m}\, \overline{x} x^{-n+1} \; (n \geqq 2). \end{aligned}\right\} \quad . \quad . \quad (7)$$

Furthermore we introduce a second system of CARTESIAN coordinates y' z', which can be derived from the y, z-system by shifting it kb to the right, and define with respect to this y' z'-system a set of stress functions $F_0^k, F_n^k, F_n^{*k}, \overline{F}_n^{*k}$ similar to (7), by replacing in these formulae x by $x' = y' + iz'$ and $\overline{x}'$ by $\overline{x}' = y' - iz'$. The representation of these functions with respect to the original coordinate-system is then obtained with the aid of the transformation-formulae: $y' = y - kb$ and $z' = z$ resp. $x' = x - kb$ and $\overline{x}' = \overline{x} - kb$. As we intend to introduce in all points $y = kb$ of the y-axis (k = integer) a same set of stress functions in order to create a periodical state of stress, and to represent all these functions with respect to the fixed system yz, it is suitable to define the new set of functions U_0, U_n, $\overline{U}_n$, U_n^*, $\overline{U}_n^*$

$$\left.\begin{aligned} &U_0 = F_0 + \sum_{k=1}^{\infty}(F_0^k + F_0^{-k}),\; U_n = F_n + \sum_{n=1}^{\infty}(F_n^k + F_n^{-k}),\; \overline{U}_n = \overline{F}_n + \sum_{n=1}^{\infty}(\overline{F}_n^k + \overline{F}_n^{-k})\;(1 \geqq 1), \\ &U_n^* = F_n^* + \sum_{k=1}^{\infty}(F_n^{*k} + F_n^{*-k}),\; \overline{U}_n^* = \overline{F}_n^* + \sum_{k=1}^{\infty}(\overline{F}_n^{*k} + \overline{F}_n^{*-k})\;(n \geqq 2). \end{aligned}\right\} \quad (8)$$

each of which represents all functions F with equal lower index n, distributed over the points kb of the y-axis.

From the way in which the functions U have been derived it follows that the function:

$$F = A_0 U_0 + A_1 U_1 + \overline{A}_1 \overline{U}_1 + \sum_{n=1}^{\infty} (A_n U_n + B_n U_n^* + \overline{A}_n \overline{U}_n + \overline{B}_n \overline{U}_n^*) \quad (9)$$

represents all possible states of stress, which can be produced by arbitrary but similar equilibrium-stress-systems acting upon the circular hole-boundaries.

We now make it our task to expand the stresses, which can be calculated from the functions (8), in FOURIER series of the argument φ. As we only wish to occupy ourselves with such stress-systems, which are symmetrical with respect to the z-axis (and consequently with respect to every z'-axis connected to $y = kb$) we restrict ourselves to the functions $U_0, U_{2n}, \overline{U}_{2n+1}, U_{2n+1}^*, \overline{U}_{2n+1}^*$ which, as a fact, do possess the required symmetry. Bearing in mind the prescribed rule of transformation, and replacing the index n by s, we find

$$\left.\begin{aligned}
F_0^k &= \Re e \left[ln\,(-kb) + ln \left(1 - \frac{x}{kb}\right)\right], \\
F_{2s}^k &= \frac{1}{(kb)^{2s}}\, \Re e \left(1 - \frac{x}{kb}\right)^{-2s} \quad (s \geqq 1), \\
\overline{F}_{2s+1}^k &= \frac{1}{(kb)^{2s+1}}\, \Im m \left(1 - \frac{x}{kb}\right)^{-(2s+1)} \quad (s \geqq 0), \\
F_{2s}^{*k} &= \frac{1}{(kb)^{2s-2}}\, \Re e \left(1 - \frac{\overline{x}}{kb}\right)\left(1 - \frac{x}{kb}\right)^{-2s+1}, \\
\overline{F}_{2s-1}^{*k} &= \frac{1}{(kb)^{2s-1}}\, \Im m \left(1 - \frac{\overline{x}}{kb}\right)\left(1 - \frac{x}{kb}\right)^{-2s} \quad (s \geqq 1).
\end{aligned}\right\} \quad . \quad (10)$$

The term $ln\ (-kb)$ may be suppressed, because it does not enter into the stress-components, all of which are defined as differential-quotients of our stress functions.

If then $ln\left(1 - \frac{x}{kb}\right)$ and all expressions $\left(1 - \frac{x}{kb}\right)^{-2s}$ and $\left(1 - \frac{x}{kb}\right)^{-2s+1}$ are expanded in power-series we get

$$F_0^k = -\Re e \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{k^n b^n},$$

$$\left.\begin{aligned}
F_{2s}^k &= \frac{1}{k^{2s}\, b^{2s}}\, \Re e \left[1 + \sum_{n=1}^{\infty} \binom{2s+n-1}{n} \frac{x^n}{k^n b^n}\right] \quad (s \geqq 1), \\
\overline{F}_{2s+1}^k &= \frac{1}{k^{2s+1}\, b^{2s+1}}\, \Im m \left[1 + \sum_{n=1}^{\infty} \binom{2s+n}{n} \frac{x^n}{k^n b^n}\right] \quad (s \geqq 0), \\
F_{2s}^{*k} &= \frac{1}{k^{2s-2}\, b^{2s-2}}\, \Re e \left(1 - \frac{\overline{x}}{kb}\right)\left[1 + \sum_{n=1}^{\infty} \binom{2s+n-2}{n} \frac{x^n}{k^n b^n}\right], \\
\overline{F}_{2s+1}^{*k} &= \frac{1}{k^{2s-1}\, b^{2s-1}}\, \Im m \left(1 - \frac{\overline{x}}{kb}\right)\left[1 + \sum_{n=1}^{\infty} \binom{2s+n-1}{n} \frac{x^n}{k^n b^n}\right] (s \geqq 1)
\end{aligned}\right\} (11)$$

from which the expressions for F_n^{-k} a.s.o. can be derived by replacing k by $-k$.

The summation over k, which now must be performed to calculate the functions U is facilitated by putting

$$\sigma_i = \sum_{k=1}^{\infty} \frac{1}{k^i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

When unessential constants are again suppressed we find:

$$\left.\begin{aligned}
U_0 &= \ln r - \sum_{n=1}^{\infty} \frac{1}{n} \sigma_{2n} \left(\frac{r}{b}\right)^{2n} \cos 2n\varphi, \\
U_{2s} &= r^{-2s} \cos 2s\varphi + \frac{1}{b^{2s}} \sum_{n=1}^{\infty} 2 \binom{2n+2s-1}{2n} \sigma_{2n+2s} \left(\frac{r}{b}\right)^{2n} \cos 2n\varphi, \quad (s \geqq 1), \\
\overline{U}_{2s+1} &= r^{-(2s+1)} \sin(2s+1)\varphi + \frac{1}{b^{2s+1}} \sum_{n=0}^{\infty} 2 \binom{2n+2s+1}{2n+1} \sigma_{2n+2s+2} \sin(2n+1)\varphi \quad (s \geqq 0), \\
U^*_{2s} &= r^{-2s+2} \cos 2s\varphi - \\
&- \frac{1}{b^{2s-2}} \left[2(2s-1)\sigma_{2s} \frac{r^2}{b^2} - \sum_{n=1}^{\infty} \left\{ 2 \binom{2n+2s-2}{2n} \sigma_{2n+2s-2} - 2 \binom{2n+2s-1}{2n+1} \sigma_{2n+2s} \frac{r^2}{b^2} \right\} \left(\frac{r}{b}\right)^{2n} \cos 2n\varphi \right] \\
\overline{U}^*_{2s+1} &= r^{-2s+1} \sin(2s+1)\varphi + \frac{1}{b^{2s-1}} \left[2(2s+1)\sigma_{2s} \frac{r}{b} \sin\varphi - 2s(2s+1)\sigma_{2s+2} \left(\frac{r}{b}\right)^3 \sin\varphi + \right. \\
&\left. + \sum_{n=1}^{\infty} \left\{ 2 \binom{2n+2s}{2n+1} \sigma_{2n+2s} - 2 \binom{2n+2s+1}{2n+2} \sigma_{2n+2s+2} \frac{r^2}{b^2} \right\} \left(\frac{r}{b}\right)^{2n+1} \sin(2n+1)\varphi \right] \quad (s \geqq 1).
\end{aligned}\right\} \quad (13)$$

As to the convergence of these series, it may be stated that the series (11) converge within the circle $|x| < kb$; therefore the convergence in (13) is limited by the smallest value of k, viz. $k = 1$, so that the radius of convergence of the series (13) is $r = b$.

Though, with the aid of (2, 4) it would be an easy matter to derive from (13) the corresponding sets of stresses — which as far as U_0, U_{2s}, U^*_{2s} are concerned, are identical with (A, 9, 14), (A, 9, 15), (A, 9, 16) — it recommends itself, in order to establish a full analogy with the treatment of the functions F discussed in the beginning of this par., to introduce a new system of elementary stress functions, analogous to (6), viz.:

$$\left.\begin{gathered}
U_{\sigma 0} \equiv a^2 U_0, \quad \overline{U}_{\sigma 1} - \overline{U}_{\tau 1} \equiv -\tfrac{1}{2} a^3 \overline{U}_1, \\
U_{\sigma,2s} \equiv -\tfrac{1}{2} a^2 \left[\frac{a^{2s-2}}{2s-1} U^*_{2s} - \frac{a^{2s}}{2s+1} U_{2s} \right], \; \overline{U}_{\sigma,2s+1} \equiv -\tfrac{1}{2} a^2 \left[\frac{a^{2s-1}}{2s} \overline{U}^*_{2s+1} - \frac{a^{2s+1}}{2s+2} \overline{U}_{2s+1} \right], \\
U_{\tau,2s} \equiv +\tfrac{1}{2} a^2 \left[\frac{a^{2s-2}}{2s-1} U^*_{2s} - \frac{(2s+2)^{2s}}{2s(2s+1)} U_{2s} \right], \\
\overline{U}_{\tau,2s+1} \equiv -\tfrac{1}{2} a^2 \left[\frac{a^{2s-1}}{2s} \overline{U}^*_{2s+1} - \frac{(2s+3)\, a^{2s+1}}{(2s+1)(2s+2)} \overline{U}_{2s+1} \right] \quad (s \geqq 1).
\end{gathered}\right\} \quad (14)$$

The stresses, caused by *these* functions at the boundary $r = a$ of all circular holes are

$$\left.\begin{array}{l}
U_{\sigma 0}\left\{\begin{array}{ll}
\sigma_r = 1 + h_0^0 & + \sum\limits_{n=1}^{\infty} h_{2n}^0 \cos 2n\varphi, \\
\sigma_\varphi = -1 + h_0^0 & + \sum\limits_{n=1}^{\infty} (h_{2n}^0 + 2\dot{j}_{2n}^0) \cos 2n\varphi, \\
\tau_{r\varphi} = & \sum\limits_{n=1}^{\infty} \dot{j}_{2n}^0 \sin 2n\varphi,
\end{array}\right. \\
\begin{array}{c} U_{\sigma,2s} \\ s \geqq 1 \end{array}\left\{\begin{array}{ll}
\sigma_r = \cos 2s\varphi + h_0^{2s} & + \sum\limits_{n=1}^{\infty} h_{2n}^{2s} \cos 2n\varphi, \\
\sigma_\varphi = \cos 2s\varphi + h_0^{2s} & + \sum\limits_{n=1}^{\infty} (h_{2n}^{2s} + 2\dot{j}_{2n}^{2s}) \cos 2n\varphi, \\
\tau_{r\varphi} = & \sum\limits_{n=1}^{\infty} \dot{j}_{2n}^{2s} \sin 2n\varphi,
\end{array}\right. \\
\begin{array}{c} U_{\tau,2s} \\ s \geqq 1 \end{array}\left\{\begin{array}{ll}
\sigma_r = & \dot{i}_0^{2s} + \sum\limits_{n=1}^{\infty} \dot{i}_{2n}^{2s} \cos 2n\varphi, \\
\sigma_\varphi = -2\cos 2s\varphi + \dot{i}_0^{2s} & + \sum\limits_{n=1}^{\infty} (\dot{i}_{2n}^{2s} + 2k_{2n}^{2s}) \cos 2n\varphi, \\
\tau_{r\varphi} = +\sin 2s\varphi \quad + & \sum\limits_{n=1}^{\infty} k_{2n}^{2s} \sin 2n\varphi.
\end{array}\right.
\end{array}\right\}. \qquad (15)$$

$$\left.\begin{array}{l}
\begin{array}{c} \overline{U}_{\sigma,2s+1} \\ s > 0 \end{array}\left\{\begin{array}{ll}
\sigma_r = \sin(2s+1)\varphi & + \sum\limits_{n=0}^{\infty} h_{2n+1}^{2s+1} \sin(2n+1)\varphi, \\
\sigma_\varphi = \sin(2s+1)\varphi & + \sum\limits_{n=0}^{\infty} (h_{2n+1}^{2s+1} + 2\dot{j}_{2n+1}^{2s+1}) \sin(2n+1)\varphi, \\
\tau_{r\varphi} = & - \sum\limits_{n=0}^{\infty} \dot{j}_{2n+1}^{2s+1} \cos(2n+1)\varphi
\end{array}\right. \\
\begin{array}{c} \overline{U}_{\tau,2s+1} \\ s > 0 \end{array}\left\{\begin{array}{ll}
\sigma_r = & - \sum\limits_{n=0}^{\infty} \dot{i}_{2n+1}^{2s+1} \sin(2n+1)\varphi, \\
\sigma_\varphi = 2\sin(2s+1)\varphi & - \sum\limits_{n=0}^{\infty} (\dot{i}_{2n+1}^{2s+1} + 2k_{2n+1}^{2s+1}) \sin(2n+1)\varphi, \\
\tau_{r\varphi} = \cos(2s+1)\varphi & + \sum\limits_{n=0}^{\infty} k_{2n+1}^{2s+1} \cos 2n+1)\varphi.
\end{array}\right.
\end{array}\right\} \qquad (16)$$

The functions $\overline{U}_{\sigma 1}$ and $\overline{U}_{\tau 1}$ only appear combined as $\overline{U}_{\sigma 1} - \overline{U}_{\tau 1}$. It must be emphasized that all stress formulae (15) and (16) contain a "main" term (identical with the stress caused by that function $F_{\sigma,2s}$, $F_{\tau,2s}$, $\overline{F}_{\sigma,2s+1}$, $\overline{F}_{\tau,2s+1}$ which corresponds to $U_{\sigma,2s}$, $U_{\tau,2s}$, $\overline{U}_{\sigma,2s+1}$, $\overline{U}_{\tau,2s+1}$) and an infinitie series of minor terms, representing the influence of the other function F of which $U_{\sigma,2s}$, $U_{\tau,2s}$, $\overline{U}_{\sigma,2s+1}$, $\overline{U}_{\tau,2s+1}$ are made up. The coefficients of these minor terms are given by (comp. (A, 9, 3) and A, 11, 4))

$$\left.\begin{aligned} h_0^{2s} &= 2\,\alpha_0^{2s}\,\sigma_{2s}\,\lambda^{2s}, \quad i_0^{2s} = 2\,\beta_0^{2s}\,\sigma_{2s}\,\lambda^{2s}, \\ h_n^s &= 2\,\alpha_n^s\,\sigma_{n+s}\,\lambda^{n+s} + 2\,\varepsilon_n^s\,\sigma_{n+s-2}\,\lambda^{n+s-2}, \quad i_n^s = 2\,\beta_n^s\,\sigma_{n+s}\,\lambda^{n+s} - 2\,\varepsilon_n^s\,\sigma_{n+s-2}\,\lambda^{n+s-2}, \\ j_n^s &= 2\,\gamma_n^s\,\sigma_{n+s}\,\lambda^{n+s} - 2\,\varepsilon_n^s\,\sigma_{n+s-2}\,\lambda^{n+s-2}, \quad k_n^s = 2\,\delta_n^s\,\sigma_{n+s}\,\lambda^{n+s} + 2\,\varepsilon_n^s\,\sigma_{n+s-2}\,\lambda^{n+s-2} \end{aligned}\right\} \quad (17)$$

For our purpose they only need to be defined for even values $(n+s)$. The quantity λ occurring in (17) defines the geometrical configuration of the holes, and is given by

$$\lambda = \frac{a}{b}, \quad \ldots\ldots\ldots\ldots \quad (18)$$

whereas the constants α_n^s, β_n^s, γ_n^s, δ_n^s, ε_n^s are defined by (comp. (A, 5, 15))

$$\left.\begin{aligned} &\alpha_0^0 = 0, \quad \alpha_n^0 = (-1)^n\,(n-1), \quad \gamma_n^0 = -\,\alpha_n^0, \\ &\alpha_0^1 + \beta_0^1 = 0, \; \alpha_n^1 + \beta_n^1 = \tfrac{1}{2}\,(-1)^{n+1}\,n\,(n-1), \quad \gamma_n^1 + \delta_n^1 = -\,(\alpha_n^1 + \beta_n^1), \\ &\alpha_0^s = (-1)^s, \quad \alpha_n^s = -\,\tfrac{1}{2}\,(-1)^{n+s}\binom{n+s}{n}\left(\frac{n\,s}{s+1} - \frac{2\,s}{n+s}\right), \\ &\qquad\qquad \gamma_n^s = +\,\tfrac{1}{2}\,(-1)^{n+s}\binom{n+s}{n}\frac{n\,s}{s+1}, \\ &\beta_0^s = -\,(-1)^s, \; \beta_n^s = +\,\tfrac{1}{2}\,(-1)^{n+s}\binom{n+s}{n}\left(\frac{n\,s}{s+1} + \frac{2\,n\,(n-1)}{(n+s)\,(s+1)} - \frac{2\,s}{n+s}\right), \\ &\delta_n^s = -\,\tfrac{1}{2}\,(-1)^{n+s}\binom{n+s}{n}\left(\frac{n\,s}{s+1} + \frac{2\,n\,(n-1)}{(n+s)\,(s+1)}\right) \quad (n \geqq 1,\; s \geqq 2), \\ &\varepsilon_n^0 = \varepsilon_n^1 = \varepsilon_1^s = 0, \; \varepsilon_n^s = \tfrac{1}{2}\,(-1)^{n+s}\,(n-1)\binom{n+s-2}{n-1} \quad (n \geqq 2,\; s \geqq 2), \end{aligned}\right\} \quad (19)$$

The numerical values (19) are to be found in (A, 5) for $s \leqq 10$ and $n \leqq 14$; the sums σ_{2i} have been calculated in 6 decimals in table 1 of (A, 9); the expressions $2\alpha_{2n}^{2s}\;\sigma_{2n+2s}$ a.s.o. have been tabulated for a sufficient number of values n and s in table 2 of (A, 9) and table 1 of (A, 11).

If finally the stress-functions (14) are combined to

$$F = C_0\,U_{\tau 0} + \sum_{s=1}^{\infty}\,(C_{2s}\,U_{\tau,2s} + D_{2s}\,U_{\tau,2s}) + \sum_{s=0}^{\infty}\,(\overline{C}_{2s+1})\,\overline{U}_{\tau,2s+1} + \overline{D}_{2s+1}\,\overline{\overline{U}}_{\tau,2s+1}), \quad (20)$$

we get a stress function which is essentially equivalent with (9) and therefore represents all possible states of stress which can be produced by arbitrary but similar equilibrium stress systems acting upon the circular hole-boundaries.

(To be continued.)

Mathematics. — *Stark konvergente Entwicklungen für die Funktionen $D(k)$ und $C(k)$.*
Von S. C. VAN VEEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

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(*Weiter zitiert als V.E. I; I u.s.w.*)

Einleitung.

Bei vielen technischen Anwendungen, insbesondere bei der Berechnung der Induktivitäten von runden Spulen, stösst man auf die folgenden vollständigen elliptischen Integrale: ¹)

$$D(k)=\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi\,d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

und

$$C(k)=\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi\cos^2\varphi\,d\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Die Integrale $D(k)$ und $C(k)$ können ohne Schwierigkeit durch die vollständigen elliptischen Integrale erster und zweiter Art ausgedruckt werden, denn man findet sofort

$$D(k)=\frac{K(k)-E(k)}{k^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

und

$$C(k)=\frac{2D(k)-K(k)}{k^2}=\frac{2\{K(k)-E(k)\}-k^2 . K(k)}{k^4} \quad . \quad . \quad . \quad (2')$$

Obgleich man also bei den hierzu gehörigen Zahlenrechnungen die klassischen Legendreschen Tafeln für $K(k)$ und $E(k)$ verwenden kann, sind doch für kleines k die so gefundenen Ergebnisse ungenau durch die Tatsache, dass im Nenner k^2 und k^4 auftritt, während im Zähler die Differenz zweier Zahlen derselben Grössenordnung auftritt. So findet man z.B. für $k=0{,}2$ aus den Tafeln von JAHNKE und EMDE

$$k^2=0{,}04; \qquad K(0{,}2)=1{,}5869; \qquad E(0{,}2)=1{,}5550$$

$$C(0{,}2)=\frac{2 . 0{,}0319-0{,}04 . 1{,}5869}{0{,}0016}=\frac{0{,}0638-0{,}0635}{0{,}0016}=0{,}1875$$

während der genaue Wert 0,20243 beträgt (J. & E. S. 82).

Diese Tatsache ist besonders von EMDE hervorgehoben worden. Es ist somit wünschens-

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¹) Vgl. K. F. MÜLLER, Arch. Elektrotechn. **17**, 336 (1926); H. NAGAOKA, Phil. Mag. **51**, 377 (1921); F. OLLENDORFF, Potentialfelder der Elektrotechnik, S. 100—123. Berlin, Springer 1922; und Zahlenbeispiel am Ende dieser Arbeit.

wert, direkte Entwicklungen für $D(k)$ und $C(k)$ abzuleiten, die diesen Uebelstand nicht zeigen. Für kleines k ist es sehr einfach, eine hypergeometrische Entwicklung für $D(k)$ und $C(k)$ nach Potenzen von k^2 abzuleiten. Die Konvergenz der bisher bekannten Entwicklungen ist aber ziemlich schwach, und schon für $k^2 = 0{,}5$ wird die Rechnung langwierig.

Für k im Mittelgebiet $0{,}5 < k < 0{,}9$ wird von EMDE die Anwendung der stark konvergenten Theta-Reihen empfohlen.

Hauptzweck der vorliegenden Arbeit ist die Ableitung einfach gebildeter hypergeometrischer Reihen, deren Konvergenzstärke nicht nur die der Theta-Reihen gleichkommt, sondern diese beliebig weit übersteigen kann.

Im § 2 wird eine Reihe exakter Entwicklungen dieser Art abgeleitet, in steigender Konvergenzstärke, u.a.

$$D(k) = \frac{\pi}{(1+\sqrt{\cos\alpha})^2}\left\{\left(1+\frac{tg^2\frac{\alpha}{2}}{2}\right) . F\left(\tfrac{1}{2},\tfrac{1}{2};1;\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\right) + \tfrac{1}{4}\,tg^2\frac{\alpha}{2}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^2 . F\left(\tfrac{1}{2},\tfrac{3}{2};2;\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\right)\right\}; \qquad (D\,III)$$

$$C(k) = \frac{\pi}{16\cos^6\frac{\alpha}{2}} . F\left(\tfrac{1}{2},\tfrac{3}{2};2;tg^4\frac{\alpha}{2}\right); \qquad (C\,II)$$

$$C(k) = \frac{\pi}{4\cos^4\frac{\alpha}{2} . (1+\sqrt{\cos\alpha})^2}\left\{F\left(\tfrac{1}{2},\tfrac{1}{2};1;\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\right) + \tfrac{1}{2}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^2 . F\left(\tfrac{1}{2},\tfrac{3}{2};2;\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\right)\right\} \qquad (C\,III)$$

u.s.w., wo $\sin\alpha = k$. Besonders die Entwicklung (C II) ist bemerkenswert. Diese Reihen konvergieren sehr stark für kleines k; ihre Konvergenz hört auf für $k = 1$. Sie können aber noch sehr gut für ziemlich grosses k angewendet werden, z.B. für $k = 0{,}9$ ist

$$\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4 = 0{,}0017524.$$

Im § 3 werden sehr einfache und genaue Approximationsformeln für $D(k)$ und $C(k)$ abgeleitet, u.a.

$$D(k) \approx \frac{\pi}{4a_3}\left(1+\frac{k_2}{2}+\frac{k_2 k_3}{4}\right); \qquad (D^*\,III)$$

$$C(k) \approx \frac{\pi}{16\,a_2^2\,a_3}\left(1+\frac{k_3}{2}\right); \qquad (C^*\,III)$$

(über die Bedeutung der Grössen α_l und k_m, vgl. V.E. I; I, Proc. XLIV, S. 967).

Die Fehler dieser Approximationsformeln werden sehr genau abgeschätzt.

Der Fehler in (D^* III) beträgt $\approx k_4 D(k)$ $(\leqq 1{,}4 \,.\, 10^{-5}$ für $\alpha \leqq 45^\circ)$.
in (C^* III) $\approx k_4 C(k)$ $(\leqq 4{,}5 \,.\, 10^{-6}$ „ „ „ „).

Wenn k in der Nähe von 1 liegt, können die Formeln für $K(k)$ und $E(k)$ aus V.E. I. verwendet werden. Die Genauigkeit wird dann nicht wesentlich durch k^2 und k^4 im Nenner verkleinert.

§ 1. *Anwendung der Landenschen Transformation.*

Satz: Für ganzes $n \geqq 1$ ist

$$K(k_1)-E(k_1)=2^{n-1}\,a_n\{K(k_n)-E(k_n)\}+\frac{k_1\,.\,K(k_n)}{a_n}\,.\sum_{m=1}^{n-1}\prod_{p=1}^{m}\left(\frac{k_p}{2}\right). \quad . \quad . \quad (3)$$

Beweis: Nach der Landenschen Transformation für die elliptischen Integrale zweiter Art ist für ganzes $n \geqq 1$

$$2\,E\,(k_n)=\frac{a_{n-1}}{a_n}\,E\,(k_{n-1})+\frac{b_{n-1}}{a_n}\,K\,(k_{n-1})\,. \quad . \quad . \quad . \quad . \quad (4)$$

(Hilfssatz II, V.E. I, III.; Proc. XLIV, S. 1198), also

$$\left.\begin{aligned}&2^n\,a_n\,\{E\,(k_n)-K\,(k_n)\}-2^{n-1}\,a_{n-1}\{E\,(k_{n-1})-K\,(k_{n-1})\}=\\&\quad=2^{n-1}\,b_{n-1}\,K\,(k_{n-1})-2^n\,a_n\,K\,(k_n)+2^{n-1}\,a_{n-1}\,K\,(k_{n-1})\end{aligned}\right\}. \quad (5)$$

Nach der Landenschen Transformation für elliptische Integrale erster Art ist für ganzes $n \geqq 1$

$$K(k_1)=\frac{K\,(k_n)}{a_n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(Satz I, V.E. I; I. Proc. XLIV, S. 967).

Aus (5) und (6) ergibt sich

$$\left.\begin{aligned}&2^n\,a_n\,\{E\,(k_n)-K\,(k_n)\}-2^{n-1}\,a_{n-1}\,\{E\,(k_{n-1})-K\,(k_{n-1})\}=\\&\quad=2^{n-1}\,K\,(k_1)\,(a_{n-1}\,b_{n-1}-2\,a_n^2+a_{n-1}^2)=\\&\quad=2^{n-1}\,K\,(k_1)\left(a_{n-1}\,b_{n-1}-\frac{a_{n-1}^2+b_{n-1}^2+2\,a_{n-1}\,b_{n-1}}{2}+a_{n-1}^2\right)=\\&\quad=2^{n-2}\,(a_{n-1}^2-b_{n-1}^2)\,K\,(k_1)\,, \qquad \text{(nach S. 967, l.c. (1))}\end{aligned}\right\}. \quad (7)$$

Weiter ist (S. 968, (8))

$$a_{m-1}^2-b_{m-1}^2=a_{m-1}^2\left(1-\frac{b_{m-1}^2}{a_{m-1}^2}\right)=a_{m-1}^2\,k_{m-1}^2\,. \quad . \quad . \quad . \quad . \quad (8)$$

Wegen (6) und (11) S. 968,

$$1+k_n=\frac{a_{n-1}}{a_n}\,; \qquad k_{n-1}=\frac{2\sqrt{k_n}}{1+k_n}$$

ist

$$k_{n-1}\,a_{n-1}=2\,a_n\sqrt{k_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

also

$$2^{2m-2}\,\frac{a_m^2}{a_1^2}=\prod_{p=2}^{m}\frac{2^2\,a_p^2}{a_{p-1}^2}=\prod_{p=2}^{m}\frac{k_{p-1}^2}{k_p}=\frac{k_1}{k_m}\prod_{p=1}^{m-1}k_p\,,$$

somit

$$a_{m-1}^2\,k_{m-1}^2=\frac{k_1}{2^{m-3}}\prod_{p=0}^{m-1}\left(\frac{k_p}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Aus (7), (8) und (10) ergibt sich

$$2^n\,a_n\{E\,(k_n)-K\,(k_n)\}-2^{n-1}\,a_{n-1}\,\{E\,(k_{n-1})-K\,(k_{n-1})\}=2\,k_1\,.\,K(k_1)\,.\prod_{p=1}^{n-1}\left(\frac{k_p}{2}\right).$$

Endlich ist

$$\begin{aligned}&2^n\,a_n\,\{E\,(k_n)-K\,(k_n)\}-2\,a_1\,\{E\,(k_1)-K\,(k_1)\}=\\&\quad=\sum_{m=2}^{n}\{2^m\,a_m\,(E\,(k_m)-K\,(k_m))-2^{m-1}\,a_{m-1}\,(E\,(k_{m-1})-K\,(k_{m-1}))\}=\\&\quad=2\,k_1\,K(k_1)\sum_{m=2}^{n}\prod_{p=1}^{m-1}\left(\frac{k_p}{2}\right)=\frac{2\,k_1\,K(k_n)}{a_n}\sum_{m=2}^{n}\prod_{p=1}^{m-1}\left(\frac{k_p}{2}\right)\\&\quad=\frac{2\,k_1\,K\,(k_n)}{a_n}\sum_{m=1}^{n-1}\prod_{p=1}^{m}\left(\frac{k_p}{2}\right) \qquad \text{nach (6).}\end{aligned}$$

Wegen $a_1=1$ ist hiermit (3) bewiesen.

$$K(k_n) = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) = \frac{\pi}{2} \prod_{p=0}^{\infty} \left(\frac{1.3.5\ldots(2p-1)}{2.4.6\ldots 2p} \right)^2 k_n^{2p},$$

$$E(k_n) = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) = \frac{\pi}{2} \prod_{p=0}^{\infty} \frac{(-1.1.3.5\ldots(2p-3))\,(1.3.5\ldots(2p-1))}{(2.4.6\ldots 2p)^2}\, k_n^{2p},$$

also

$$K(k_n) - E(k_n) = \frac{\pi}{2} k_n^2 \prod_{p=0}^{\infty} \left(\frac{1.3.5\ldots(2p-1)}{2.4.6\ldots 2p} \right)^2 \left(\frac{2p+1}{2p+2} \right) k_n^{2p} = \frac{\pi}{4} k_n^2\, F(\tfrac{1}{2}, \tfrac{3}{2}; 2; k_n^2). \qquad (11)$$

Aus (3) und (11) ergibt sich, wegen (10)

$$\left.\begin{aligned} K(k_1) - E(k_1) &= 2^{n-3} a_n k_n^2 \pi F(\tfrac{1}{2}, \tfrac{3}{2}; 2; k_n^2) + \frac{\pi k_1}{2 a_n} \sum_{m=1}^{n-1} \prod_{p=1}^{m} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) = \\ &= \frac{\pi k_1}{2 a_n} \left\{ \sum_{m=1}^{n-1} \prod_{p=1}^{m} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) + \prod_{p=1}^{n} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{3}{2}; 2; k_n^2) \right\} \quad (n \geqq 1) \end{aligned}\right\} (12)$$

§ 2. *Exakte stark konvergente Reihenentwicklungen für die Funktionen D(k) und C(k), wenn k nicht in der Nähe von 1 liegt.*

Nach (1′) und (12) ist

$$D(k_1) = \frac{K(k_1) - E(k_1)}{k_1^2} = \frac{\pi}{2 a_n k_1} \left\{ \sum_{m=1}^{n-1} \prod_{p=1}^{m} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) + \prod_{p=1}^{n} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{3}{2}; 2; k_n^2) \right\}, \qquad (13)$$

$(n \geqq 1)$ und nach (2′) und (12) ist

$$\left.\begin{aligned} C(k_1) &= \frac{2D(k_1) - K(k_1)}{k_1^2} = \frac{2}{k_1^2} D(k_1) - \frac{1}{a_n k_1^2} K(k_n) = \\ &= \frac{\pi}{2 a_n k_1^2} \left\{ \sum_{m=2}^{n-1} \prod_{p=2}^{m} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) + \prod_{p=2}^{n} \left(\frac{k_p}{2} \right) F(\tfrac{1}{2}, \tfrac{3}{2}; 2; k_n^2) \right\} \quad (n \geqq 2) \\ &= \frac{\pi k_2}{4 a_n k_1^2} \left\{ \sum_{m=2}^{n-1} \prod_{p=3}^{m} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) + \prod_{p=3}^{n} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{3}{2}; 2; k_n^2) \right\} \\ &= \frac{\pi}{16 a_2^2 a_n} \left\{ \sum_{m=2}^{n-1} \prod_{p=3}^{m} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k_n^2) + \prod_{p=3}^{n} \left(\frac{k_p}{2} \right) . F(\tfrac{1}{2}, \tfrac{3}{2}; 2; k_n^2) \right\}, (n \geqq 3) \end{aligned}\right\} (14)$$

(wegen (9)). Aus (13) findet man, wegen V.E. I, I, S. 967, (3) und S. 968, (5) wenn man suzzessive $n = 1, 2, 3$ setzt

$$D(k_1) = \frac{\pi}{4} F(\tfrac{1}{2}, \tfrac{3}{2}; 2; \sin^2 \alpha); \qquad (D\,I)$$

$$D(k_1) = \frac{\pi}{4 \cos^2 \frac{\alpha}{2}} \left\{ F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; tg^4 \frac{\alpha}{2} \right) + \frac{tg^2 \frac{\alpha}{2}}{2} . F\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; tg^4 \frac{\alpha}{2} \right) \right\}; \qquad (D\,II)$$

$$\begin{aligned} D(k_1) = \frac{\pi}{(1+\sqrt{\cos\alpha})^2} \Bigg\{ & \left(1 + \frac{tg^2 \frac{\alpha}{2}}{2} \right) . F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}} \right)^4 \right) + \\ & + \frac{tg^2 \frac{\alpha}{2}}{4} \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}} \right)^2 . F\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}} \right)^4 \right) \Bigg\}; \qquad (D\,III) \end{aligned}$$

$$D(k_1)=\frac{\pi}{\left(\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}\right)^2}\left\{\left(1+\frac{tg^2\frac{\alpha}{2}}{2}+\frac{tg^2\frac{\alpha}{2}}{4}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^2\right).F\left(\tfrac{1}{2},\tfrac{1}{2};1;\left(\frac{\cos\frac{\alpha}{2}-\sqrt[4]{\cos\alpha}}{\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}}\right)^4\right)\right.$$

$$\left.+\frac{tg^2\frac{\alpha}{2}}{8}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^2\left(\frac{\cos\frac{\alpha}{2}-\sqrt[4]{\cos\alpha}}{\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}}\right)^2.F\left(\tfrac{1}{2},\tfrac{3}{2};2;\left(\frac{\cos\frac{\alpha}{2}-\sqrt[4]{\cos\alpha}}{\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}}\right)^4\right)\right\};\quad (D\,IV).$$

u. s. w.

Ebenso findet man aus (14), wenn man $n=2$, 3 und 4 wählt,

$$C(k_1)=\frac{\pi}{16\cos^6\frac{\alpha}{2}}.F\left(\tfrac{1}{2},\tfrac{3}{2};2;tg^4\frac{\alpha}{2}\right);\qquad (C.\,II)$$

$$C(k_1)=\frac{\pi}{4\cos^4\frac{\alpha}{2}.(1+\sqrt{\cos\alpha})^2}\left\{F\left(\tfrac{1}{2},\tfrac{1}{2};1;\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\right)+\right.$$

$$\left.+\tfrac{1}{2}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^2.F\left(\tfrac{1}{2},\tfrac{3}{2};2;\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\right)\right\};\quad (C.\,III)$$

$$C(k_1)=\frac{\pi}{4\cos^4\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}\right)^2}\left\{\left(1+\tfrac{1}{2}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^2\right)F\left(\tfrac{1}{2},\tfrac{1}{2};1;\left(\frac{\cos\frac{\alpha}{2}-\sqrt[4]{\cos\alpha}}{\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}}\right)^4\right)+\right.$$

$$\left.+\tfrac{1}{4}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^2\left(\frac{\cos\frac{\alpha}{2}-\sqrt[4]{\cos\alpha}}{\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}}\right)^2.F\left(\tfrac{1}{2},\tfrac{3}{2};2;\left(\frac{\cos\frac{\alpha}{2}-\sqrt[4]{\cos\alpha}}{\cos\frac{\alpha}{2}+\sqrt[4]{\cos\alpha}}\right)^2\right)\right\};\quad (C.\,IV)$$

u. s. w.

Weil für kleine Werte von α $k_n \approx \frac{\sin^{2^n-1}\alpha}{2^{2^n-2}}$, konvergieren die vorhergehenden hypergeometrischen Reihen ausserordentlich stark, (insbes. III und IV), selbst dann, wenn $\frac{\pi}{2}-\alpha$ nicht *sehr* klein ist, wie aus der folgenden Tabelle erhellt:

α	$k_1=\sin\alpha$	k_3^2	k_4^2	k_5^2
70°	0,93969	$4{,}7.10^{-3}$	$1{,}4.10^{-6}$	
85°	0,99619	$8{,}8.10^{-2}$	$5{,}3.10^{-4}$	$1{,}7.10^{-8}$
89°	0,99985	$3{,}5.10^{-1}$	$1{,}1.10^{-2}$	$2{,}8.10^{-3}$

(*A*)

Für $k=1$ ($\alpha=90°$) hört die Konvergenz der Reihen auf, doch für $\alpha=89°$ sind die Reihen IV und V noch sehr gut brauchbar.

§ 3. *Approximationsformeln für die Funktionen D(k) und C(k), wenn k nicht in der Nähe von 1 liegt.*

Aus der Tabelle (*A*) geht hervor, dass k_n^2 mit n sehr stark abnimmt.

Mit Vernachlässigung von k_n^2 findet man aus (13) und (14) die sehr einfach gebildeten Approximationsformeln [1])

---

[1]) Ein leeres Produkt, wie z.B. $\prod\limits_{p=2}^{1}\left(\frac{k_p}{2}\right)$ wird $=1$ gesetzt.

$$D(k_1)\approx D_n(k_1)=\frac{\pi}{2a_n k_1}\sum_{m=1}^{n}\prod_{p=2}^{m}\left(\frac{k_p}{2}\right)=\frac{\pi}{4a_n}\sum_{m=1}^{n}\prod_{p=2}^{m}\left(\frac{k_p}{2}\right);\quad (n\geqq 1),\quad (15)$$

$$C(k_1)\approx C_n(k_1)=\frac{\pi}{2a_n k_1^2}\sum_{m=2}^{n}\prod_{p=2}^{m}\left(\frac{k_p}{2}\right)=\frac{\pi}{16a_2^2 a_n}\sum_{m=2}^{n}\prod_{p=3}^{m}\left(\frac{k_p}{2}\right);\quad (n\geqq 2).\quad (16)$$

§ 4. *Abschätzung der Fehler der Approximationsformeln.*

Obgleich es nicht schwer ist, die Fehler dieser Approximationsformeln in direkter Weise abzuschätzen, gewinnt man in viel einfacherer Weise Ausdrücke für die Differenzen $D_{n+1}(k_1)-D_n(k_1)$, bzw. $C_{n+1}(k_1)-C_n(k_1)$, die mit starker Annäherung die Grösse des Fehlers in $D_n(k_1)$ bzw. $C_n(k_1)$ liefern.

Man findet aus (15)

$$D_{n+1}(k_1)-D_n(k_1)=\frac{\pi}{4}\sum_{m=1}^{n}\prod_{p=2}^{m}\left(\frac{k_p}{2}\right)\left(\frac{1}{a_{n+1}}-\frac{1}{a_n}\right)+\frac{\pi}{4a_{n+1}}\prod_{p=2}^{n+1}\left(\frac{k_p}{2}\right)$$

$$=\frac{\pi}{4a_n}\sum_{m=1}^{n}\prod_{p=2}^{m}\left(\frac{k_p}{2}\right)\left(\frac{a_n}{a_{n+1}}-1\right)+\frac{\pi}{2^{n+2}a_{n+1}k_1}\cdot\prod_{p=1}^{n+1}k_p$$

$$=k_{n+1}D_n(k_1)+\frac{2^{n-2}\pi\,\alpha_{n+1}k_{n+1}^2}{k_1^2}\quad \text{(wegen V. E. I; I; (6). und (10) dieser Arbeit)}$$

Also ist:

$$D_{n+1}(k_1)-D_n(k_1)=k_{n+1}\left(D_n(k_1)+\frac{2^{n-2}\pi\,a_{n+1}k_{n+1}}{k_1^2}\right).\quad ,\quad (17)$$

Ebenso findet man aus (16)

$$\left.\begin{aligned}C_{n+1}(k_1)-C_n(k_1)&=\frac{\pi}{16a_2^2}\sum_{m=2}^{n}\prod_{p=3}^{m}\left(\frac{k_p}{2}\right)\left(\frac{1}{a_{n+1}}-\frac{1}{a_n}\right)+\frac{\pi}{16a_2^2 a_{n+1}}\prod_{p=3}^{n+1}\left(\frac{k_p}{2}\right)=\\&=k_{n+1}\left(C_n(k_1)+\frac{2^{n-1}\pi\,.\,a_{n+1}k_{n+1}}{k_1^4}\right)\end{aligned}\right\}.\quad (18)$$

Für nicht zu kleines n $(n\geqq 3)$ kann, wenn k_1 nicht in der Nähe von 1 liegt, kann $\frac{2^{n-2}\pi a_{n+1}\cdot k_{n+1}}{k_1^2}$ in (17) gegen $D_n(k_1)$ und $\frac{2^{n-1}\pi a_{n+1}k_{n+1}}{k_1^4}$ in (18) gegen $C_n(k_1)$ vernachlässigt werden, weil k_{n+1} stark abnimmt bei wachsendem n. Man findet somit für $n\geqq 3$, $\alpha<80^\circ$

$$D(k_1)-D_n(k_1)\approx D_{n+1}(k_1)-D_n(k_1)\approx k_{n+1}D_n(k_1)\quad .\ .\quad (17')$$

$$C(k_1)-C_n(k_1)\approx C_{n+1}(k_1)-C_n(k_1)\approx k_{n+1}C_n(k_1).\quad .\ .\quad (18')$$

Für $n=3$ findet man die einfaches Sonderfälle

$$D(k_1)\approx D_3(k_1)=\frac{\pi}{4a_3}\left(1+\frac{k_2}{2}+\frac{k_2k_3}{4}\right)\qquad (D^*\,III)$$

worin der Fehler

$\approx k_4 D_3(k_1)$ $\quad < 1{,}4\,.\,10^{-5}$ für $\alpha\leqq 45^\circ$ und $<12\,.\,10^{-3}$ für $\alpha\leqq 80^\circ$

und

$$C(k_1)\approx C_3(k_1)=\frac{\pi}{16a_2^2a_3}\left(1+\frac{k_2}{2}\right),\qquad (C^*\,III)$$

worin der Fehler

$\approx k_4 C_3(k_1)$ $\quad <4{,}5\,.\,10^{-6}$ für $\alpha\leqq 45^\circ$ und $<6\,.\,10^{-3}$ für $\alpha\leqq 80^\circ$.

Wählt man $n = 4$, so bekommt man

$$D(k_1) \approx D_4(k_1) = \frac{\pi}{4\,a_4}\left(1 + \frac{k_2}{2} + \frac{k_2 k_3}{4} + \frac{k_2 k_3 k_4}{8}\right), \qquad (D^* IV)$$

worin der Fehler

$\approx k_5\, D_4(k_1) \qquad < 4{,}9 \,.\, 10^{-11}$ für $\alpha \leqq 45^\circ$ und $< 2{,}2 \,.\, 10^{-5}$ für $\alpha \leqq 80^\circ$,

und schliesslich

$$C(k_1) \approx C_4(k_1) = \frac{\pi}{16\,a_2^2\,a_4}\left(1 + \frac{k_3}{2} + \frac{k_3 k_4}{4}\right), \qquad (C^* IV)$$

worin der Fehler

$\approx k_5\, C_5(k_1) \qquad < 1{,}6 \,.\, 10^{-11}$ für $\alpha \leqq 45^\circ$ und $< 10^{-5}$ für $\alpha \leqq 80^\circ$.

Die Formeln D^* und C^* sind somit ausserordentlich genau im Gebiet $\alpha \leqq 45^\circ$.

§ 5. *Einige Bemerkungen über die Rechnung, wenn k in der Nähe von 1 liegt.*

Wenn die Differenz $1 - k$ sehr klein ist, kann man in (1′) und (2′) besser die Entwicklungen (35) V.E. I; II (Proc. XLIV, S. 1082) und (47) V.E. I; III (Proc. XLIV, S. 1203) anwenden, nämlich

$$K(k_1) = \frac{1}{2^{q-1}c_q}\log\frac{4}{l_q}\,.\,F(\tfrac{1}{2}, \tfrac{1}{2}; 1; l_q^2) - \frac{1}{2^{q-2}c_q\,.\,\pi}\,.\sum_{m=1}^{\infty}\frac{1}{(2m-1)2m}\,.\sum_{n=m}^{\infty}\left\{\frac{\Gamma(\frac{1}{2}+n)}{n!}\right\}^2 l_q^{2n}$$

und

$$E(k_1) = c_n F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; l_n^2) + \frac{c_n}{2}\, l_n^2 \log\frac{4}{l_n}\,.\,F(\tfrac{1}{2}, \tfrac{3}{2}; 2; l_n^2)$$

$$-\frac{2c_n}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)(2m+2)}\sum_{q=m}^{\infty}\frac{\Gamma(q+\frac{1}{2})\,.\,\Gamma(q+\frac{3}{2})}{q!\,(q+1)!}\, l_n^{2q+2} + l_1\,.\,K(k_1)\sum_{m=2}^{n}\prod_{p=1}^{m-1}\left(\frac{l_p}{2}\right).$$

Die Approximationsformeln sind $\left(\text{mit Vernachlässigung von } l_n^2 \log\frac{4}{l_n}\right)$

$$K_n(k_1) \approx \frac{1}{2^{n-1}c_n}\log\frac{4}{l_n};$$

$$E_n(k_1) \approx c_n + \frac{l_1}{2^{n-1}c_n}\,.\log\frac{4}{l_n}\sum_{m=1}^{n}\prod_{p=1}^{m}\left(\frac{l_p}{2}\right).\ ^{1)}$$

Näheres hierüber folgt in einer grösseren Arbeit in der „Zeitschrift für angew. Math. und Mech."

§ 6. *Vorschriften zur numerischen Rechnung. Zahlenbeispiel.*

Setzt man $k_n = \sin\alpha_n \quad (\alpha_1 = \alpha)$, so ist

$$\sin\alpha_{n+1} = k_{n+1} = \frac{1 - \sqrt{1 - k_n^2}}{1 + \sqrt{1 - k_n^2}} \qquad \text{(Proc. XLIV, S. 968, (9))}$$

$$= \frac{1 - \cos\alpha_n}{1 + \cos\alpha_n} = tg^2\,\frac{\alpha_n}{2}.$$

Hieraus können die Grössen k_n sehr einfach logarithmisch berechnet werden.

---

1) Diese Formel ist in Proc. XLIV, S. 1205 (β) fehlerhaft angegeben worden. Man lese dort in den letzten zwei Zeilen:

$$E(k_1) \approx c_n + \frac{1}{2^{n-1}c_n}\log\frac{4}{l_n}\sum_{m=2}^{n+1}2^{m-1}\, l_m\, c_m^2 = c_n + \frac{l_1}{2^{n-1}c_n}\log\frac{4}{l_n}\sum_{m=2}^{n}\frac{l_1\,.\,l_2\ldots l_m}{2^m}.$$

Wegen (10) ist

$$a_n = \frac{\sqrt{k_1 \prod\limits_{p=1}^{n} k_p}}{2^{n-1}\, k_n}.$$

Für kleines k ist oft die Reihenentwicklung

$$k_{n+1} = \frac{1-\sqrt{1-k_n^2}}{1+\sqrt{1-k_n^2}} = \frac{k_n^2}{4}\, F(1, \tfrac{3}{2}; 3; k_n^2) = \frac{k_n^2}{4}\,(1 + \tfrac{1}{2}\, k_n^2 + \tfrac{5}{16}\, k_n^4 + \ldots)$$

nützlich.

Zahlenbeispiel:

Gegeninduktivität M zweier paralleler koaxialer Kreisströme.

Kreisradien a und A, Abstand der Mittelpunkte h. Dann ist [1])

$$M = \frac{4\pi}{k}\sqrt{A\,.\,a}\,.\,\{(2-k^2)K(k) - 2E(k)\} = 4\pi\sqrt{Aa}\,\Big\{\frac{2}{k}(K-E) - kK\Big\} = 4\pi k^3\sqrt{Aa}\,.\,C(k)$$

mit

$$k^2 = \frac{4Aa}{(A+a)^2 + h^2}.$$

Man findet also:

$$\frac{M}{4\pi\sqrt{Aa}} = k^3\, C(k).$$

Wir wählen den ziemlich grossen Wert $k = \sin 80° = 0.9848$.

$$C_4(k) = \frac{\pi}{16\, a_2^2\, a_4}\left(1 + \frac{k_3}{2} + \frac{k_3\, k_4}{2}\right).$$

$$\begin{array}{l|l}
\log k_1 = \log \sin \alpha_1 = 9{,}9933515 & k_1 = 0{,}9848080 \\
\log k_2 = 2 \log tg \frac{\alpha_1}{2} = 9{,}8476270 = \log \sin \alpha_2;\ \alpha_2 = 44°45'21'',31 & k_2 = 0{,}7040880 \\
\log k_3 = 2 \log tg \frac{\alpha_2}{2} = 9{,}2292048 = \log \sin \alpha_3;\ \alpha_3 = 9°45'34'',36 & k_3 = 0{,}1695137 \\
\log k_4 = 2 \log tg \frac{\alpha_3}{2} = 7{,}8626576 = \log \sin \alpha_4;\ \alpha_4 = 25'\ \ 3'',44 & k_4 = 0{,}007288826
\end{array}$$

$$\begin{array}{rl}
\log k_1 = & \underline{9{,}9933515}\ + \\
6{,}9261924 - 10 : 2 = & 8{,}4630962 - 10 \\
\log k_4 = & \underline{7{,}8626576 - 10}\ - \\
\log 8\, a_4 = & 0{,}6004386 \\
2 \log a_2 = 4 \log \cos 40° = & 9{,}5370160 \\
\log 2 = & \underline{0{,}3010300}\ + \\
\log 16\, a_2^2\, a_4 = & 0{,}4384846
\end{array}$$

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[1]) Vgl. z.B. MAXWELL: Treatise on Electricity and Magnetism. 2nd Ed. Oxford, 1881, II, p. 310.

$$\frac{k_3}{2} = 0{,}08475685$$

$$\frac{k_3 k_4}{4} = 0{,}00030889$$

$$1 + \frac{k_3}{2} + \frac{k_3 k_4}{4} = 1{,}08506574 \quad \rightarrow \quad \log = 0{,}0354561$$

$$\log \pi = 0{,}4971499$$

$$3 \log k_1 = 9{,}9800545 \quad +$$

$$0{,}5126605$$

$$\log 16\, a_2^2\, a_4 = 0{,}4384846 \quad -$$

$$\log k_1^3\, C_4(k_1) = \underline{0{,}0741759}$$

$$k_1^3\, C_4(k_1) = 1{,}186249$$

$$\text{Fehler} \approx k_1^3\, k_5\, C_4(k_1) \approx k_1^3\, C_4(k_1) \,.\, \frac{k_4^2}{4} = 0{,}000016 \quad +$$

$$k_1^3\, C_5(k_1) = \underline{1{,}186265}$$

$$\log k_1^3\, C_5(k_1) = \underline{0{,}0741816}$$

in genauer Uebereinstimmung mit MAXWELL (l.c. S. 319).

Mathematics. — *Conformal differential geometry.* II. *Curves in conformal two-dimensional spaces.* By J. HAANTJES. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of February 28, 1942.)

Summary.

In a former paper[1]) a method has been introduced for developing the conformal differential geometry of curves in flat spaces of dimension $n > 2$. In this note it is proved that the same theory holds also for $n = 2$ if we restrict ourselves to the conformal transformations of the Möbius group. In particular the conformal Frenet-Serret formulae, which give differential relations between the fundamental quantities of a curve, have exactly the same form. Furthermore geometrical interpretations are given of these fundamental quantities, which include among other things the conformal parameter and the inversion curvature.

The fundamental theorem.

Let $a_{\lambda\varkappa}$ be the fundamental tensor of a 2-dimensional flat space R_2, in which the coordinates are denoted by $x^{\varkappa}$. This coordinate system is assumed to be a rectangular cartesian one, though we need not to restrict ourselves to these systems. In C.D.G. I[1]) we have proved the following theorem:

The conformal invariant properties in a flat space are those properties, which are invariant under a conformal transformation of the fundamental tensor

$$a'_{\lambda\varkappa} = \sigma^2 a_{\lambda\varkappa}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

such that the space remains a flat space.

Therefore, the curvature $\varkappa'$ of the metric tensor $a'_{\lambda\varkappa}$ has to vanish. This curvature is given by the equation

$$\varkappa' = - \sigma^{-2} a^{\mu\nu} \partial_\mu \partial_\nu \log \sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

taken from SCHOUTEN–STRUIK[2]). Hence the function σ in (1) must satisfy the equation

$$a^{\mu\nu} \partial_\mu s_\nu = 0 ; \quad s_\nu = \partial_\nu \log \sigma . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The above theorem applies to the whole set of conformal transformations and it enables us to develop the differential geometry of this set of transformations.

In this paper however we will restrict ourselves to those conformal transformations, which transform circles into circles (the so called Möbius group). This restriction imposes an additional condition on σ, which can be deduced by requiring that a circle remains a circle, if $a'_{\lambda\varkappa}$ is taken as the fundamental tensor instead of $a_{\lambda\varkappa}$.

Let the arc-lengths of a curve C with respect to $a_{\lambda\varkappa}$ and $a'_{\lambda\varkappa}$ be denoted by s and s', the corresponding covariant derivatives along the curve by δ/ds and δ'/ds' respectively. The coordinate system being a cartesian one for the metric $a_{\lambda\varkappa}$, the covariant derivative δ/ds is identical with the ordinary derivative d/ds. If $i^{\varkappa}$ is the unit vector tangent to the curve, the curvature k of C may be found from the Frenet formulae

$$\frac{\delta i^{\varkappa}}{ds} = k \underset{1}{i}{}^{\varkappa} ; \quad \frac{\delta \underset{1}{i}{}^{\varkappa}}{ds} = - k i^{\varkappa}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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[1]) Conformal differential geometry. Curves in conformal euclidean spaces, Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 814—824 (1941), referred to as C.D.G. I.

[2]) SCHOUTEN–STRUIK, Einführung in die neueren Methoden der Differentialgeometrie. II (Noordhoff, 1938) p. 291, formel (19. 1 β).

where $\underset{1}{i}^{\varkappa}$ is a unit vector normal to $i^{\varkappa}$. Since $i^{\varkappa}$ and $\underset{1}{i}^{\varkappa}$ are unit vectors, they transform under a conformal transformation (1) of the $a_{\lambda\varkappa}$ as follows

$$i'^{\varkappa} = \sigma^{-1} i^{\varkappa}; \quad \underset{1}{i}'^{\varkappa} = \sigma^{-1} \underset{1}{i}^{\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These transformed vectors are related by the following differential equation

$$\frac{\delta' i'^{\varkappa}}{ds'} = k' \underset{1}{i}'^{\varkappa}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where k' is the curvature of C with respect to the tensor $a'_{\lambda\varkappa}$. Now if $p^{\varkappa}$ is any contravariant vector we have [3])

$$\frac{\delta' p^{\varkappa}}{ds'} = \sigma^{-1} \left\{ \frac{\delta p^{\varkappa}}{ds} + (s_{\mu} i^{\mu}) p^{\varkappa} + (s_{\mu} p^{\mu}) i^{\varkappa} - a_{\mu\lambda} p^{\mu} i^{\lambda} s^{\varkappa} \right\}. \quad . \quad . \quad . \quad (7)$$

From the equations (4), (6) and (7) follows by a simple calculation the relation between k and k'

$$k' = \sigma^{-1} (k - s_{\mu} \underset{1}{i}^{\mu}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence we have

$$\frac{dk'}{ds'} = \sigma^{-1} \left\{ \frac{dk}{ds} - (\partial_{\nu} s_{\mu}) i^{\nu} \underset{1}{i}^{\mu} + s_{\mu} s_{\nu} i^{\mu} \underset{1}{i}^{\nu} \right\}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now suppose the curve C is a circle with respect to the metric $a_{\lambda\varkappa}$. Then k is constant and so will be k' if a circle will remain a circle under the conformal transformation of the fundamental tensor. This leads as is seen from (9) to the vanishing of

$$(\partial_{\nu} s_{\mu} - s_{\nu} s_{\mu}) i^{\nu} \underset{1}{i}^{\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

for every pair of two mutually orthogonal vectors i^{ν} and $\underset{1}{i}^{\mu}$, which condition is equivalent with the equation

$$\partial_{\nu} s_{\mu} - s_{\nu} s_{\mu} = \lambda a_{\mu\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Multiplication by $a^{\mu\nu}$ gives in connection with (3)

$$\lambda = -\tfrac{1}{2} a^{\mu\nu} s_{\nu} s_{\mu} = -\tfrac{1}{2} s_{\nu} s^{\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Hence σ satisfies the equation

$$\partial_{\nu} s_{\mu} - s_{\nu} s_{\mu} + \tfrac{1}{2} a_{\mu\nu} s_{\varrho} s^{\varrho} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

As (3) is a consequence of (13), this equation is the only condition imposed upon σ. So we have arrived at the following result.

The conformal properties, which are invariant under the transformations of the Möbius group, are those properties, which are invariant under a conformal transformation of the fundamental tensor $a'_{\lambda\varkappa} = \sigma^2 a_{\lambda\varkappa}$, *$\sigma$ satisfying the equation* (13).

By comparing this theorem with the result obtained in C.D.G. I concerning the conformal transformations in an R_n $(n > 2)$, we see that for every $n > 1$ σ satisfies the same equation. But only for $n > 2$ the equation (13) appears to be a direct consequence of the vanishing of the curvature affinor belonging to $a'_{\lambda\varkappa}$. As a result of this the conformal theory developed in C.D.G. I may be applied to the case $n = 2$. We give here a brief summary of the results specialised for $n = 2$.

---

[3]) Compare C.D.G. I, p. 816.

Fundamental relations.

a. Let the curvatures of a curve for the metrics $a_{\lambda\varkappa}$ and $a'_{\lambda\varkappa}$ be denoted by k and k' respectively. From (9) and (13) it follows that

$$\frac{dk'}{ds'} = \sigma^{-2} \frac{dk}{ds} . \qquad (14)$$

We choose the direction of increasing s so that $\frac{dk}{ds}$ is positive. The relation (14) enables us to define on the curve a conformal invariant parameter τ

$$\tau = \int \sqrt{\varrho}\, ds + \text{constant};\ \varrho = \frac{dk}{ds} . \qquad (15)$$

This parameter of the third order is called the *conformal parameter* of the curve.

b. Instead of $i^\varkappa$ and $\underset{1}{i}^\varkappa$ (comp. (4) we use the following conformal invariant vectors

$$j^\varkappa = \frac{dx^\varkappa}{d\tau} = \varrho^{-\frac{1}{2}} i^\varkappa ;\ \underset{1}{j}^\varkappa = \varrho^{-\frac{1}{2}} \underset{1}{i}^\varkappa , \qquad (16)$$

which have the direction of the tangent and the normal respectively.

c. The covariant differentiation to s being not a conformal invariant differentiation is replaced by a *conformal covariant differentiation* to the parameter τ. This differentiation is defined by the connexion parameters

$$\Gamma^\varkappa_{\mu\lambda} = \left\{ {\varkappa \atop \mu\lambda} \right\} + Q_\mu A^\varkappa_\lambda + Q_\lambda A^\varkappa_\mu - a_{\mu\lambda} a^{\varkappa\nu} Q_\nu , \qquad (17)$$

where $A^\varkappa_\lambda$ is the unit affinor and Q_μ is given by the equation

$$Q_\mu = k \underset{1}{i}_\mu + \tfrac{1}{2} \left(\frac{d}{ds} \log \varrho \right) i_\mu . \qquad (18)$$

The transformation of Q_μ under conformal transformations is given by

$$Q'_\mu = Q_\mu - s_\mu , \qquad (19)$$

from which it follows at once that the parameters $\Gamma^\varkappa_{\mu\lambda}$ are invariant. The conformal covariant derivative to the parameter τ is denoted by the symbol D_τ.

d. The conformal "Frenet-Serret" formulae for the curve are

$$\left.\begin{aligned} & D_\tau j^\varkappa \equiv \frac{dj^\varkappa}{d\tau} + \Gamma^\varkappa_{\mu\lambda} j^\mu j^\lambda = 0 \\ & D_\tau \underset{1}{j}^\varkappa = 0 \\ & D_\tau Q_\lambda + (j^\mu Q_\mu) Q_\lambda - \tfrac{1}{2} Q_\mu Q^\mu j_\lambda = (j^\mu j_\mu)^{-1} (h j_\lambda + \underset{1}{j}_\lambda) . \end{aligned}\right\} \qquad (20)$$

e. The function $h(\tau)$ is a conformal invariant of the fifth order of the curve. It is called the *inversion curvature* of the curve [4]. The function $h(\tau)$ determines the curve to within conformal representations belonging to the Möbius group. So the equation of the curve may be written in the form

$$h = h(\tau) . \qquad (21)$$

This equation is called its *intrinsic equation.*

---

[4] BLASCHKE uses the invariant $b = 2h$. (Vorlesungen über Differentialgeometrie, III).

f. An important theorem concerning touching curves is the following one. A necessary and sufficient condition in order that two curves have at a point P at least a five-point (six-point) contact is that the quantities $j^\varkappa$, Q_μ (and h) at P be the same for both curves. This theorem is an immediate consequence of the definition of the quantities involved.

The loxodromic.

An isogonal trajectory of a system of circles passing through two fixed points is called a loxodromic. In order to find the intrinsic equation of a loxodromic we choose the fundamental tensor $a_{\lambda\varkappa}$ so that the family of circles with respect to the metric $a_{\lambda\varkappa}$ is a pencil of straight lines through a fixed point P. If the constant angle under which the curve $x^\varkappa = x^\varkappa(s)$ meets the lines is denoted by α, the coordinates of P are given by

$$y^\varkappa = x^\varkappa + \lambda \cos\alpha\, i^\varkappa + \lambda \sin\alpha\, \underset{1}{i}{}^\varkappa. \quad \ldots\ldots \quad (22)$$

By differentiating (22) we obtain the following two equations for λ and k, the curvature of the loxodromic,

$$\frac{d\lambda}{ds} = -\cos\alpha, \qquad \lambda k = \sin\alpha \quad \ldots\ldots \quad (23)$$

from which we get

$$\lambda = -s\cos\alpha, \quad k = -\frac{tg\,\alpha}{s}, \quad \frac{dk}{ds} = \frac{tg\,\alpha}{s^2} = \varrho \text{ 5)}. \quad \ldots \quad (24)$$

Consequently

$$Q_\mu = -\frac{tg\,\alpha}{s}\, \underset{1}{i}{}_\mu - \frac{1}{s}\, i_\mu. \quad \ldots\ldots \quad (25)$$

Then the invariant h can be calculated from (20). It appears to be

$$h = cotg\, 2\alpha. \quad \ldots\ldots \quad (26)$$

Hence we have the theorem

The curves of constant inversion curvature are loxodromics.

Consider a point $x^\varkappa$ of the loxodromic. Besides the osculating circle at $x^\varkappa$ there exist several other circles connected with the curve at this point f.i. the circle through $x^\varkappa$ belonging to the coaxial system by which the loxodromic is defined and the circle through $x^\varkappa$ normal to this system.

The center of the circle through the point $x^\varkappa$ orthogonal to the pencil is $y^\varkappa$. Hence its curvature with respect to $a_{\lambda\varkappa}$ is

$$|\lambda|^{-1} = \frac{1}{s\cos\alpha} = Q_\mu\,(-\sin\alpha\, \underset{1}{i}{}^\mu - \cos\alpha\, i^\mu) = Q_\mu\, v^\mu, \quad \ldots \quad (27)$$

where v^μ is the unit vector pointing towards the center of the circle. This equation however is conformal invariant as may be seen from (8) and (19). So (27) gives the curvature with respect to any metric tensor obtained from $a_{\lambda\varkappa}$ by a conformal transformation.

The circle through $x^\varkappa$ belonging to the pencil by which the loxodromic is defined is a straight line with respect to the fundamental tensor $a_{\lambda\varkappa}$. Its direction is given by the vector

$$\sin\alpha\, \underset{1}{i}{}^\varkappa + \cos\alpha\, i^\varkappa. \quad \ldots\ldots \quad (28)$$

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5) Here $tg\,\alpha$ is supposed to be positive.

Its curvature c with respect to the metric tensor $a_{\lambda\varkappa}$ vanishes. With respect to any other metric tensor obtained from $a_{\lambda\varkappa}$ by a conformal transformation the curvature is given by the following invariant equation

$$c = Q_\mu (\cos a\, \underset{1}{i}^{\mu} - \sin a\, i^\mu) = Q_\mu w^\mu, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where w^μ is the unit vector normal to the tangent of the circle. Indeed, this expression vanishes if Q_μ is replaced by the value (25), whereas both sides of (29) transform under conformal transformations in the same way as a consequence of (8) and (19), w^μ being a unit vector.

We shall use these results in the following section.

Geometrical interpretations of τ, Q_μ and h.

The parameter τ. If d denotes the distance between the centers of two circles C_1 and C_2 with radius r_1 and r_2, the expression

$$I = \sqrt{\frac{(r_1 - r_2)^2 - d^2}{2\, r_1\, r_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

is a conformal invariant of the curve. This invariant I is a function of the cross-ratio of the points of intersection of the two circles with an arbitrarily chosen circle orthogonal to C_1 and C_2 [6]). When the expression (30) is calculated for the osculating circles of the curves at the points s and $s + \triangle s$, we obtain

$$\triangle I = \frac{dk}{ds} \triangle s^2 + \ldots . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Consequently we have from (15) to within terms of higher order

$$\triangle \tau = \sqrt{\triangle I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

which gives us at the same time a geometrical interpretation of τ.

The quantity Q_μ. Before we turn to the geometrical interpretation of Q_μ, we ask for the geometrical figure, which corresponds to a covariant vector w_μ at a point $P(x_0^\varkappa)$, transforming under conformal transformations as follows

$$w'_\mu = w_\mu - s_\mu . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Let $e^\varkappa$ be any contravariant unit vector at P. Then the transformation of the scalar $p = e^\mu w_\mu$ is given by

$$p' = \sigma^{-1} (p - e^\mu s_\mu) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

By comparing (34) with (8) we see that this transformation is identical with that of the curvature of a circle through P, whose tangent at P is orthogonal to e^μ. Therefore e^μ and w_μ together define a circle through P with the center

$$x^\varkappa = x_0^\varkappa + p^{-1} e^\varkappa . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

In varying the unit vector $e^\varkappa$ we obtain a family of ∞^1 circles all passing through the point P. It is seen from (35) that the locus of the centers of these circles is a straight line given by

$$w_\lambda (x^\lambda - x_0^\lambda) = 1 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Hence this family of circles is a coaxial system, whose axis (36) is orthogonal to the vector $w^\varkappa$.

Now the transformation of Q_μ is identical with that of w_μ. Hence we have the theorem

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6) Comp. W. BLASCHKE, Vorlesungen über Differentialgeometrie, III, p. 41.

The covariant vector Q_μ corresponds geometrically to a coaxial system of circles.

Henceforth this system of circles will be denoted by (Q). It may be noted that the curvature of the circle of the system (Q), which is tangent to the curve at the point under consideration is given by

$$Q_\mu \underset{1}{i}^\mu = k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

from which it follows that this circle is identical with the osculating circle of the curve. Hence the pencil (Q) contains the osculating circle. So its axis passes through the center of curvature and is normal to the vector $Q^\varkappa$.

In the following a geometrical property will be given of this particular system of circles, which leads at the same time to a geometrical interpretation of Q_μ.

Consider a loxodromic having at P at least a five-point contact with the given curve C. Then as we have seen the quantities $j^\varkappa$, $\underset{1}{j}^\varkappa$ and Q_μ at P are the same for both the curve and the loxodromic. If besides that the inversion curvature of both curves are equal we have at P a six-point contact. Now a curve is determined by the values of the quantities $j^\varkappa$, $\underset{1}{j}^\varkappa$ and Q_μ at one point together with the function $h(\tau)$ [7], which is a constant for a loxodromic. Therefore, there exists only one loxodromic, which has at P a six-point contact with C and a system of ∞^1 loxodromics, which have at P at least a five-point contact with the curve C. Each of these ∞^1 loxodromics meets a coaxial system of circles under a constant angle α, which is connected by the inversion curvature of the loxodromic by formel (26). Now consider the circle through P normal to one of these coaxial systems. Its curvature is according to (27) given by

$$Q_\mu (-\sin\alpha\, \underset{1}{i}^\mu - \cos\alpha\, i^\mu) = Q_\mu v^\mu, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

its center by

$$y^\varkappa = x^\varkappa + (Q_\mu v^\mu)^{-1} v^\varkappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

from which it follows that this circle belongs to the system (Q) at P. But to every value of α corresponds according to (26) one value of h, thus one loxodromic having at P a five-point contact with the curve. Hence each circle of the system (Q) can be obtained in this way. This result enables us to state the following theorem.

There exists a family of ∞^1 loxodromics having at least a five-point contact with a given curve at a point P. Each of these loxodromics meets a pencil of circles under a constant angle. The system of circles through P each of which is normal to one of these pencils form the coaxial system (Q) at P.

Another geometrical interpretation of the pencil (Q) is obtained as follows. Consider again a loxodromic which has at P a five-point contact with the curve, together with the coaxial system of circles, which are cut by this loxodromic under a constant angle α. The center of the circle through P belonging to this system is according to (28) and (29) given by

$$x^\varkappa + (Q_\mu w^\mu)^{-1} w^\varkappa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

where

$$w^\varkappa = \cos\alpha\, \underset{1}{i}^\varkappa - \sin\alpha\, i^\varkappa. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

Hence this circle too belongs to the system (Q) at P. If we had started with another loxodromic having a five-point contact with the curve, we should have obtained another circle of (Q). We may state this result thus:

Of each pencil of circles belonging to a loxodromic, which has at P at least a five-point contact with a curve, one circle passes through P. These circles through P together form the coaxial system (Q) of the curve at P.

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[7] This theorem has been proved in C.D.G. I.

The invariant h. A geometrical interpretation of the inversion curvature h is obtained by considering the loxodromic, which has at P a six-point contact with the given curve and whose inversion curvature is therefore equal to that of the curve at P. We then arrive at the theorem.

The loxodromic which has at P a six-point contact with a given curve C meets a pencil of circles under a constant angle α. The inversion curvature of C at the point P is connected with this angle by the formel

$$h = \operatorname{cotg} 2\alpha. \quad \ldots \ldots \ldots \quad (42)$$

Another geometrical interpretation of the invariant h is obtained by considering the circle of the system (Q), which is normal to the curve at P. This circle is called the *normal circle* of the curve at P. In the following it will appear that if h is negative two consecutive normal circles have real points of intersection and therefore meet under a real angle. The center of the normal circle is given by

$$y^\varkappa = x^\varkappa + p\, i^\varkappa; \quad p = (Q_\mu\, i^\mu)^{-1}. \quad \ldots \ldots \quad (43)$$

Then if $\theta(\tau)$ is the angle between the normal circles of the curve at the points $P(\tau_0)$ $P'(\tau)$ we have at P

$$\left(\frac{d\theta}{d\tau}\right)^2 = \frac{1}{p^2}\left\{\Sigma\left(\frac{dy^\varkappa}{d\tau}\right)^2 - \left(\frac{dp}{d\tau}\right)^2\right\}. \quad \ldots \ldots \quad (44)$$

Since

$$\frac{dy^\varkappa}{d\tau} = i^\varkappa\left(\varrho^{-\frac{1}{2}} + \frac{dp}{d\tau}\right) + \varrho^{-\frac{1}{2}}\, k\, p\, \underset{1}{i}^\varkappa, \quad \ldots \ldots \quad (45)$$

the equation (44) reduces to

$$\left(\frac{d\theta}{d\tau}\right)^2 = \frac{\varrho^{-1}}{p^2}\left\{1 + 2\varrho^{\frac{1}{2}}\frac{dp}{d\tau} + k^2 p^2\right\}. \quad \ldots \ldots \quad (46)$$

From the fundamental formulea (20) we obtain

$$-\frac{1}{p^2}\frac{dp}{d\tau} = \varrho^{\frac{1}{2}} h + \tfrac{1}{2}\varrho^{-\frac{1}{2}}\left(k^2 + \frac{1}{p^2}\right). \quad \ldots \ldots \quad (47)$$

When this expression is substituted in (46) it is found that the inversion curvature h satisfies the equation

$$\left(\frac{d\theta}{d\tau}\right)^2 = -2h. \quad \ldots \ldots \ldots \quad (48)$$

This relation bears out the statement that θ only exists for negative h. So we have arrived at a geometrical interpretation of h, expressed by the following theorem[8])

The normal circles at the points τ and $\tau + \triangle\tau$ of a curve of negative inversion curvature meet under an angle $\triangle\theta$ for which we have to within terms of higher order

$$\triangle\theta = \sqrt{-2h}\,\triangle\tau. \quad \ldots \ldots \ldots \quad (49)$$

For curves of positive inversion curvature we obtain in much the same way

$$\left(\frac{dI}{d\tau}\right)^2 = h, \quad \ldots \ldots \ldots \quad (50)$$

where I is the conformal invariant of the two normal circles at P and P', defined by (30).

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[8]) This geometrical interpretation of h has been given by J. MAEDA, Geometrical meanings of the inversion curvature of a plane curve. Jap. J. Math. **16**, 177—232 (1940).

Mathematics. — *Sur le théorème de* MINKOWSKI, *concernant un système de formes linéaires réelles.* I. *Première communication: Introduction, Applications.* Par J. F. KOKSMA et B. MEULENBELD. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 28, 1942.)

§ 1. Soit $L_\nu = a_{\nu 1} x_1 + \dots + a_{\nu, n+1} x_{n+1}$ $(\nu = 1, \dots, n+1)$ un système de $n+1 \geqq 2$ formes linéaires à coefficients réels, et à déterminant $\Delta = |a_{\nu\mu}| \neq 0$, et soit $\tau_\nu (\nu = 1, \dots, n+1)$ un système de $n+1$ nombres positifs avec $\tau_1 \dots \tau_{n+1} = |\Delta|$. Alors d'après le théorème célèbre de MINKOWSKI [1]) il y a au moins un système de nombres entiers $(x_1, \dots, x_{n+1}) \neq (0, \dots, 0)$ tel que

$$|L_1| \leqq \tau_1, |L_2| < \tau_2, \dots, |L_{n+1}| < \tau_{n+1}, \quad \dots \dots \quad (1)$$

d'où

$$|L_1 \dots L_{n+1}| < |\Delta|. \quad \dots \dots \dots \quad (2)$$

D'après un autre théorème de MINKOWSKI [2]) pour tout nombre $\sigma \geqq 1$, fixe et positif, il y a au moins un système de nombres entiers $(x_1, \dots, x_{n+1}) \neq (0, \dots, 0)$ tel que

$$\frac{|L_1|^\sigma + \dots + |L_{n+1}|^\sigma}{n+1} \leqq B(n, \sigma) \, |\Delta|^{\frac{\sigma}{n+1}}, \quad \dots \dots \quad (3)$$

où

$$B(n, \sigma) = \left\{ \frac{(n+1)^{-\frac{n+1}{\sigma}} \, \Gamma\left(1 + \frac{n+1}{\sigma}\right)}{\Gamma^{n+1}\left(1 + \frac{1}{\sigma}\right)} \right\}^{\frac{\sigma}{n+1}}. \quad \dots \dots \quad (4)$$

Recherchant les formes quadratiques positives définites, M. H. F. BLICHFELDT [3]) a amélioré ce résultat pour $\sigma = 2$; sa preuve a été simplifiée par M. R. REMAK [4]).

M.M. J. G. VAN DER CORPUT et G. SCHAAKE [5]) ont généralisé le théorème de BLICHFELDT, concernant le cas $\sigma \geqq 2$. Ils remplacent (4) par

$$B(n, \sigma) = \tfrac{1}{2} \left\{ \frac{(n+1)^{-\frac{n+1}{\sigma}} \, \Gamma\left(1 + \frac{n+1}{\sigma}\right) (n + 1 + \sigma)}{\Gamma^{n+1}\left(1 + \frac{1}{\sigma}\right) \sigma} \right\}^{\frac{\sigma}{n+1}}, \quad \dots \quad (5)$$

résultat, qui pour $\sigma = 2$ coïncide avec celui de M. BLICHFELDT.

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[1]) H. MINKOWSKI, Geometrie der Zahlen. Leipzig—Berlin, Kap. 4 (1910). Comparer aussi: J. F. KOKSMA, Diophantische Approximationen, Berlin, p. 15 (1936).

[2]) Voir [1]). MINKOWSKI, G. d. Z., p. 122; KOKSMA, D. A., p. 22.

[3]) H. F. BLICHFELDT, A new principle in the geometry of numbers, with some applications. Trans. Amer. Math. Soc. 15, 227—235 (1914).

[4]) R. REMAK, Vereinfachung eines BLICHFELDTschen Beweises aus der Geometrie der Zahlen. Math. Z. Bd. 26, 694—699 (1927).

[5]) J. G. VAN DER CORPUT et G. SCHAAKE, Anwendung einer BLICHFELDTschen Beweismethode in der Geometrie der Zahlen. Acta Arith. 2, 152—160 (1937).

M. BLICHFELDT avec une autre méthode a amélioré son résultat encore; il trouve [6]):

$$B(n,2)=\frac{2}{\pi}\left\{\frac{(n+1)^{-\frac{n+1}{2}}\,\Gamma\left(1+\frac{n+3}{2}\right)}{1+g}\right\}^{\frac{2}{n+1}} \quad \ldots\ldots \quad (6)$$

où

$$g=\frac{1}{2^{\frac{n+1}{2}}}\left\{\frac{1}{n+2}-(2-\sqrt{2})^{n+2}\left(\frac{1}{\sqrt{2}}+\frac{1}{n+2}\right)\right\}. \quad \ldots \quad (7)$$

En appliquant le théorème de la moyenne géométrique et la moyenne arithmétique on tire de (3) l'inégalité:

$$|L_1\ldots L_{n+1}|\leqq B(n,\sigma)^{\frac{n+1}{\sigma}}\,|\Delta| \quad (\sigma\geqq 1), \quad \ldots\ldots \quad (8)$$

qui est plus exacte que (2), dès que $B(n,\sigma)<1$.

Les résultats cités admettent des applications aux approximations simultanées de n nombres réels $\alpha_1,\ldots,\alpha_n$, et à l'approximation diophantique de la forme linéaire $\alpha_1 x_1+\ldots+\alpha_n x_n - x_{n+1}$ à zéro.

Dans ce mémoire nous montrerons encore un théorème sur le système de formes de MINKOWSKI, admettant des applications analogues. C'est le

Théorème 1. *Soient n et r des nombres naturels, $1\leqq r\leqq n$; pour $\frac{n+1}{2}\leqq r\leqq n$ le nombre $\varrho_{n,r}$ soit désigné par*

$$\left.\begin{aligned}\varrho_{n,r}=2^{n+1}\Bigg\{&\frac{1}{(n+1)!\,r^r}\sum_{\mu=0}^{r}\binom{n+1}{\mu}\left(r-\frac{n+1}{2}\right)^{\mu}(n+1-r)^{r-\mu}+\\&+\frac{1}{r!}\sum_{\mu=0}^{n-r}\frac{r^{\mu+1}}{(n+1-r)^{\mu+1}(n-r-\mu)!}\left(\frac{n+1}{2r}\right)^{\frac{r(n-r-\mu)}{n+1-r}}P_\mu\Bigg\},\end{aligned}\right\} \quad (9)$$

où

$$P_\mu=\int_0^{\frac{2r-n-1}{n+1}}\frac{dv_{\mu+1}}{(v_{\mu+1}+1)^{\frac{n+1}{n+1-r}}}\int_0^{v_{\mu+1}}\frac{dv_\mu}{(v_\mu+1)^{\frac{n+1}{n+1-r}}}\cdots\int_0^{v_3}\frac{dv_2}{(v_2+1)^{\frac{n+1}{n+1-r}}}\int_0^{v_2}\frac{v^r{}_1\,dv_1}{(v_1+1)^{\frac{n+1}{n+1-r}}},$$

et pour $1\leqq r\leqq n$ le nombre $\varrho^*_{n,r}$ par

$$\left.\begin{aligned}&\varrho^*_{n,r}=\varrho_{n,r} \quad \text{pour} \quad \frac{n+1}{2}\leqq r\leqq n,\\&\varrho^*_{n,r}=\varrho_{n,n+1-r} \quad \text{pour} \quad 1\leqq r<\frac{n+1}{2}.\end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

En outre soient $L_1,\ldots,L_{n+1}$ des formes linéaires:

$$L_1=\alpha_{11}x_1+\ldots\ldots\ldots+\alpha_{1,n+1}x_{n+1}$$
$$\vdots$$
$$L_{n+1}=\alpha_{n+1,1}x_1+\ldots\ldots+\alpha_{n+1,n+1}x_{n+1}$$

à coefficients réels $\alpha_{\nu\mu}$, tels que le déterminant $\Delta=|\alpha_{\nu\mu}|$ ne s'annule pas.

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[6]) H. F. BLICHFELDT, The minimum value of quadratic forms and the closest packing of spheres. Math. Ann. Bd. **101**, 605—608 (1929).

Alors à tout nombre $t > 2$ *au moins un système de nombres entiers* $(x_1, \ldots, x_{n+1})$ *correspond satisfaisant à*

$$X = \max(|x_1|, \ldots, |x_{n+1}|) \geqq 1,$$

et aux inégalités simultanées:

$$\sum_{\nu=1}^{r} |L_\nu| \leqq 2 \sqrt[r]{\frac{t^{n+1-r}|\Delta|}{\varrho^*_{n,r}}}, \quad \ldots\ldots \quad (11)$$

$$\sum_{\nu=r+1}^{n+1} |L_\nu| \leqq \frac{2}{t}, \quad \ldots\ldots\ldots\ldots \quad (12)$$

$$\left(\sum_{\nu=1}^{r} |L_\nu|\right)^r \left(\sum_{\nu=r+1}^{n+1} |L_\nu|\right)^{n+1-r} \leqq \frac{|\Delta|}{\varrho^*_{n,r}}, \quad \ldots\ldots \quad (13)$$

$$|L_1 \ldots L_{n+1}| \leqq \frac{|\Delta|}{\varrho^*_{n,r}\, r^r\, (n+1-r)^{n+1-r}}. \quad \ldots\ldots \quad (14)$$

Nous démontrerons ce théorème à l'aide d'une méthode de M. BLICHFELDT [3]), et bien dans notre deuxième communication; la troisième et la quatrième communication de ce mémoire contiendront la démonstration d'un lemme essentiel (le lemme 1), dont nous ferons usage déjà dans la deuxième communication.

Remarques.

1. D'après le théorème 1 il existe pour tout nombre $t > 2$ au moins un système de nombres entiers $(x_1, \ldots x_{n+1})$ avec $X \geqq 1$, (11), (12), (13) et (14). Si toujours (c'est à dire pour tout système de nombres entiers $(x_1, \ldots, x_{n+1})$ avec $X \geqq 1$) $\sum_{\nu=r+1}^{n+1} |L_\nu| \neq 0$, ce système ne peut pas rester le même, quand t croît indéfiniment, comme on voit de (12) immédiatement. Il y a donc une infinité de tels systèmes $(x_1, \ldots, x_{n+1})$ avec $X \geqq 1$, (13) et (14). Dans les cas où $\sum_{\nu=1}^{r} |L_\nu| = 0$ ou $\sum_{\nu=r+1}^{n+1} |L_\nu| = 0$ cette dernière assertion est triviale, parce que avec $(x_1, \ldots, x_{n+1})$ pour tout nombre entier p aussi $(px_1, \ldots, px_{n+1})$ est une solution de (13) et (14).

2. Pour $r = n$ les nombres $\varrho_{n,r}$ et $\varrho^*_{n,r}$ définis dans le théorème 1 coïncident et nous posons

$$\varrho_n = \varrho^*_{n,n} = \varrho_{n,n} = 2^{n+1} \left\} \frac{1}{(n+1)!\, n^n} \sum_{\mu=0}^{n} \binom{n+1}{\mu} \left(n - \frac{n+1}{2}\right)^\mu + \frac{n}{n!} \int_0^{\frac{n-1}{n+1}} \frac{v^n}{(v+1)^{n+1}}\, dv \right\{$$

$$= 2^{n+1} \left\} \frac{1}{(n+1)!\, n^n} \left(1 + \frac{n-1}{2}\right)^{n+1} - \frac{1}{(n+1)!\, n^n} \left(\frac{n-1}{2}\right)^{n+1} + \frac{n}{n!} \int_0^{\frac{n-1}{2n}} \frac{w^n}{1-w}\, dw \right\{$$

$$\left(\text{nous avons substitué} \quad w = \frac{v}{1+v}\right)$$

$$= \left\} \frac{(n+1)^{n+1}}{(n+1)!\, n^n} - \frac{(n-1)^{n+1}}{(n+1)!\, n^n} + \frac{2^{n+1}\, n}{n!} \sum_{\mu=n+1}^{\infty} \frac{1}{\mu} \left(\frac{n-1}{2n}\right)^\mu \right\{$$

$$= \frac{(n+1)^n}{n!\, n^n} \left\} 1 - \frac{(n-1)^{n+1}}{(n+1)^{n+1}} + \frac{2^{n+1}\, n^{n+1}}{(n+1)^n} \sum_{\mu=n+1}^{\infty} \frac{1}{\mu} \left(\frac{n-1}{2n}\right)^\mu \right\{$$

et donc

$$\varrho_n = \frac{(n+1)^n}{n!\,n^n}\left\{1+\frac{2^{n+1}\,n^{n+1}}{(n+1)^n}\sum_{\mu=n+2}^{\infty}\frac{1}{\mu}\left(\frac{n-1}{2n}\right)^{\mu}\right\}. \quad . \quad . \quad . \quad (15)$$

§ 2. Appliquons le théorème 1 à l'approximation de la forme linéaire

$$a_1 x_1 + \ldots + a_n x_n - x_{n+1}$$

à zéro. Nous trouvons le

Théorème 2. *Soit n un nombre naturel supérieur à 1 et ϱ_n le nombre désigné par* (15). *Soient en outre $a_1, \ldots, a_n$ n nombres réels, et t un nombre arbitraire > 2.*

Alors il y a au moins un système de nombres entiers $(x_1, \ldots, x_{n+1})$ avec

$$X = max.(|x_1|, \ldots, |x_n|) \geqq 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

et

$$\sum_{\nu=1}^{n} |x_\nu| \leqq 2\sqrt[n]{\frac{t}{\varrho_n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

tel que la forme linéaire

$$L = a_1 x_1 + \ldots + a_n x_n - x_{n+1}$$

vérifie les inégalités simultanées.

$$|L| \leqq \frac{2}{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$|L| \leqq \frac{1}{\varrho_n\left(\sum\limits_{\nu=1}^{n}|x_\nu|\right)^{n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

et

$$|L| \leqq \frac{1}{\varrho_n (n-p)^n\, |x_1 \ldots x_n|^{\frac{n}{n-p}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

où p désigne le nombre des x_ν qui sont égaux à zéro, tandis que dans le membre droit de (20) *les $x_\nu = 0$ sont remplacés par $x_\nu = 1$.*

Remarques.

3. L'inégalité (20) suit de (19) immédiatement: En appliquant le théorème de la moyenne géométrique et la moyenne arithmétique on obtient de (19), c'est à dire de

$$|L|^{\frac{1}{n}} \sum_{\nu=1}^{n} |x_\nu| \leqq \frac{1}{\varrho_n^{\frac{1}{n}}}$$

l'inégalité

$$|L|^{\frac{n-p}{n}}\, |x_1 \ldots x_n| \leqq \left(\frac{1}{\varrho_n^{\frac{1}{n}}(n-p)}\right)^{n-p}$$

en remarquant que p des x_ν sont égaux à zéro. En vertu de la première remarque il y a donc une infinité de systèmes $(x_1, \ldots, x_{n+1})$ avec (16), (19) et (20).

4. La démonstration du théorème 2 nous donnerons en § 4 de cette première communication.

5. On voit que chez l'approximation de L à zéro en (19) le quantité $\sum_{\nu=1}^{n} |x_\nu|$, et en (20) le quantité $|x_1 \dots x_n|$ se présentent. Tandis que dans la littérature une déduction des approximations de L analogues à (19) ne nous est pas connue, un mémoire récent de M. N. HOFREITER[7]) s'agit des approximations de L analogues à (20): En appliquant les inégalités (8) et (4) de MINKOWSKI avec

$$L_1 = \frac{x_1}{t}, \dots, L_n = \frac{x_n}{t}, L_{n+1} = L t^n, \sigma = 1,$$

on obtient l'inégalité

$$|x_1 \dots x_n| \, |L| < \frac{n!}{(n+1)^n} \quad \dots\dots\dots \quad (21)$$

et en appliquant pour $\sigma = 2$ l'inégalité (8) avec la valeur (5) de BLICHFELDT on obtient

$$|x_1 \dots x_n| \, |L| < \left(\frac{\gamma_{n+1}}{n+1}\right)^{\frac{n+1}{2}} \quad \dots\dots\dots \quad (22)$$

où donc γ_{n+1} est un nombre satisfaisant à l'inégalité

$$\gamma_{n+1} \leqq \frac{2}{\pi} \left\{ \Gamma\left(1 + \frac{n+3}{2}\right) \right\}^{\frac{2}{n+1}} \quad \dots\dots\dots \quad (23)$$

Parce que (21) et (22) sont triviales, si une au moins des valeurs x_ν est égale à zéro, M. HOFREITER démontre que (21) et (22) possèdent une infinité de solutions entières $x_1, \dots, x_n$ avec $x_1 \dots x_n \neq 0$, mais en ce point ci sa démonstration ne nous semble pas correcte. Nous remarquons qu'on peut éviter cette difficulté en faisant usage de (3) au lieu de (8): en appliquant le théorème de la moyenne géométrique et la moyenne arithmétique et en introduisant le nombre p des x_ν qui sont égaux à zéro, on peut remplacer (21) et (22) par des formules analogues à (20). Mais ces formules présenteront le même inconvénient que (20), à savoir qu'on ignore la valeur de p.

Afin de compléter notre exposé des résultats connus sur l'approximation de L à zéro, remarquons qu'il y a encore une troisième façon d'approximer L, et bien avec $X = \max. (|x_1|, \dots, |x_n|)$ au lieu des expressions $\sum_{\nu=1}^{n} |x_\nu|$ ou $|x_1 \dots x_n|$. Dans un mémoire précédent[8]) nous avons démontré pour une infinité de systèmes entiers $(x_1, \dots, x_{n+1})$ l'inégalité:

$$|L| \leqq \frac{1}{n! \, \varrho_n X^n} \cdot \quad \dots\dots\dots \quad (24)$$

§ 3. Une deuxième application du théorème 1 nous trouvons en considérant la somme et le produit des n différences absolues, paraissant chez l'approximation simultanée d'un système de nombres réels $(\alpha_1, \dots, \alpha_n)$ par des fractions rationnelles. Nous obtenons le

Théorème 3. *Soient n un nombre naturel supérieur à 1 et ϱ_n le nombre désigné par*

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[7]) N. HOFREITER, Diophantische Approximationen komplexer Zahlen. Mh. f. Math. u. Phys. **49**, 299—302 (1940).

[8]) J. F. KOKSMA et B. MEULENBELD, Ueber die Approximation einer homogenen Linearform an die Null. Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 62—74 (1941).

(15). *Soient en outre* $(\alpha_1, \ldots, \alpha_n)$ *un système de* n *nombres réels, et* t *un nombre arbitraire* > 2.

Alors il y a au moins un système de nombres entiers $(x_1, \ldots, x_{n+1})$ *qui satisfont aux inégalités simultanées:*

$$1 \leqq |x_{n+1}| \leqq 2\,\frac{t^n}{\varrho_n}, \qquad \ldots \ldots \ldots \quad (25)$$

$$\sum_{\nu=1}^{n} \left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right| \leqq \frac{2}{t\,|x_{n+1}|}, \qquad \ldots \ldots \ldots \quad (26)$$

$$\sum_{\nu=1}^{n} \left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right| \leqq \frac{1}{\varrho_n^{\frac{1}{n}}\,|x_{n+1}|^{1+\frac{1}{n}}}, \qquad \ldots \ldots \ldots \quad (27)$$

$$\prod_{\nu=1}^{n} \left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right| \leqq \frac{1}{\varrho_n\, n^n\, |x_{n+1}|^{n+1}}. \qquad \ldots \ldots \quad (28)$$

Remarques.

6. En vertu de la première remarque il y a donc une infinité de systèmes entiers $(x_1, \ldots, x_{n+1})$ avec $|x_{n+1}| \geqq 1$, (27) et (28).

7. La démonstration du théorème 3 nous donnerons dans § 5 de cette communication.

8. Autant qu'il nous soit connu, l'approximation de la somme $\sum_{\nu=1}^{n} \left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right|$ et du produit $\prod_{\nu=1}^{n} \left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right|$ à zéro ne sont jamais considérées.

Au contraire l'approximation simultanée du système $(\alpha_1, \ldots, \alpha_n)$ par des fractions rationnelles est bien recherchée. Des résultats nous citons l'approximation de MINKOWSKI 1):

$$\left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right| \leqq \frac{1}{\left(1 + \frac{1}{n}\right) |x_{n+1}|^{1+\frac{1}{n}}} \qquad (\nu = 1, \ldots, n),$$

qui est améliorée par M. BLICHFELDT 3) à:

$$\left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right| \leqq \frac{1}{(n\,!)^{\frac{1}{n}}\, \varrho_n^{\frac{1}{n}}\, |x_{n+1}|^{1+\frac{1}{n}}} \qquad (\nu = 1, \ldots, n). \quad \ldots \quad (29)$$

En addisant les dernières inégalités nous obtenons:

$$\sum_{\nu=1}^{n} \left| \alpha_\nu - \frac{x_\nu}{x_{n+1}} \right| \leqq \frac{n}{(n\,!)^{\frac{1}{n}}\, \varrho_n^{\frac{1}{n}}\, |x_{n+1}|^{1+\frac{1}{n}}}.$$

Une comparaison de cette inégalité avec (27) nous démontre en vertu de $\frac{n}{(n\,!)^{\frac{1}{n}}} \geqq 1$, que notre résultat est une amélioration essentielle. C'est aussi le cas quand on forme le produit des inégalités (29) et compare avec (28).

En appliquant la méthode des formes quadratiques définites à

$$t\,|\alpha_1 x_{n+1} - x_1|^2 + t\,|\alpha_2 x_{n+1} - x_2|^2 + \ldots + t\,|\alpha_n x_{n+1} - x_n|^2 + \frac{x_{n+1}^2}{t^n}$$

on trouve

$$x_{n+1}^2 \prod_{\nu=1}^{n} (a_\nu x_{n+1} - x_\nu)^2 < \left(\frac{\gamma_{n+1}}{n+1}\right)^{n+1}$$

ou

$$\prod_{\nu=1}^{n} \left| a_\nu - \frac{x_\nu}{x_{n+1}} \right| < \left(\frac{\gamma_{n+1}}{n+1}\right)^{\frac{n+1}{2}} \frac{1}{|x_{n+1}|^{n+1}},$$

avec (23). Pour n très grand ce résultat est meilleur que (28).

§ 4. *Démonstration du théorème 2.*

Nous appliquons le théorème 1 avec

$$L_\nu = x_\nu \; (\nu = 1, \ldots, n); \; L_{n+1} = L; \; r = n; \; \varrho^*_{n,n} = \varrho_{n,n} = \varrho_n.$$

Alors $|\triangle| = 1$. Il existe donc au moins un système $(x_1, \ldots, x_{n+1})$ avec max. $(|x_1|, \ldots, |x_{n+1}|) \geqq 1$, (17), (18), (19) et (20). Nous remarquons qu'il est impossible que $x_\nu = 0$ pour toute valeur de $\nu = 1, \ldots, n$. Car dans ce cas nous aurions

$|L| = |x_{n+1}| \geqq 1$, ce qui est en contradiction avec (18), où $|L| \leqq \frac{2}{t} < 1$.

Alors on a $X = \max. (|x_1|, \ldots, |x_n|) \geqq 1$, et le théorème est démontré.

§ 5. *Démonstration du théorème 3.*

Nous appliquons maintenant le théorème 1 avec

$$L_1 = x_{n+1}; \; L_\nu = x_{n+1} a_{\nu-1} - x_{\nu-1} \; (\nu = 2, \ldots, n+1); \; r = 1; \; \varrho^*_{n,1} = \varrho_{n,n} = \varrho_n.$$

Alors il est encore $|\triangle| = 1$. Il existe donc au moins un système $(x_1, \ldots, x_{n+1})$ avec max. $(|x_1|, \ldots, |x_{n+1}|) \geqq 1$,

$$|x_{n+1}| \leqq \frac{2t^n}{\varrho_n}, \quad \ldots\ldots\ldots\ldots \quad (30)$$

$$\sum_{\nu=1}^{n} |x_{n+1} a_\nu - x_\nu| \leqq \frac{2}{t}, \quad \ldots\ldots\ldots\ldots \quad (31)$$

$$\sum_{\nu=1}^{n} |x_{n+1} a_\nu - x_\nu| \leqq \frac{1}{\varrho_n^{\frac{1}{n}} |x_{n+1}|^{\frac{1}{n}}}, \quad \ldots\ldots \quad (32)$$

$$\prod_{\nu=1}^{n} |x_{n+1} a_\nu - x_\nu| \leqq \frac{1}{\varrho_n n^n |x_{n+1}|}. \quad \ldots\ldots \quad (33)$$

Il est impossible que $x_{n+1} = 0$. Dans ce cas nous aurions au moins une valeur de $\nu = 1, \ldots, n$, où $x_\nu \neq 0$, donc $\sum_{\nu=1}^{n} |x_\nu| \geqq 1$, ce qui est en contradiction avec (31) où $\sum_{\nu=1}^{n} |x_\nu| \leqq \frac{2}{t} < 1$. On a donc $|x_{n+1}| \geqq 1$, et alors d'après (30), (31), (32) et (33) il suit l'assertion de ce théorème.

Mathematics. — *Contribution à la théorie métrique des approximations diophantiques non-linéaires.* (Deuxième communication.) Par J. F. KOKSMA. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 28, 1942.)

§ 1. *Introduction* [1].

I. Soit $f(1), f(2), \ldots$ une suite arbitraire de nombres naturels croissants. Par $d(z, x)$ j'indique le plus grand diviseur commun $(f(z), f(x))$ des nombres $f(z)$ et $f(x)$ $(z, x = 1, 2, \ldots)$. Alors pour toute suite de cette espèce et toute paire de nombres α, β satisfaisant aux inégalités $0 \leqq \alpha < \beta \leqq 1$, je définis la fonction arithmétique $A(x, \alpha, \beta)$ $(x = 1, 2, \ldots)$ comme le nombre des nombres naturels de l'intervalle $\alpha f(x) < u < \beta f(x)$ qui pour aucune valeur de l'indice z $(1 \leqq z < x)$ ne sont divisibles par le nombre $\frac{f(x)}{d(z, x)}$.

Soit S un système de $n \geqq 1$ suites de nombres naturels croissants

$$f_\nu(1), f_\nu(2), \ldots (\nu = 1, 2, \ldots, n), \quad \ldots \ldots \ldots \quad (1)$$

B un système de n paires de nombres α_ν, β_ν avec $0 \leqq \alpha_\nu < \beta_\nu \leqq 1$ $(\nu = 1, 2, \ldots, n)$ et soit posé pour tout nombre naturel N

$$H(N, B) = H(N, B, S) = \frac{1}{N} \sum_{x=1}^{N} \sum_{\nu=1}^{n} \frac{A_\nu(x, \alpha_\nu, \beta_\nu)}{f_\nu(x)}, \quad \ldots \quad (2)$$

où $A_\nu(x, \alpha_\nu, \beta_\nu)$ désigne la fonction $A(x, \alpha_\nu, \beta_\nu)$ correspondant à la suite $f_\nu(1), f_\nu(2), \ldots$.

Définition 3. *Nous dirons que le système S possède la propriété $\mathfrak{R}$, si au système spécial B_0 des nombres $\alpha_\nu = 0, \beta_\nu = 1$ $(\nu = 1, 2, \ldots, n)$ correspondent un nombre positif c et un indice N_0, indépendant de N, tels que*

$$H(N, B_0, S) \geqq c, \quad \text{si} \quad N \geqq N_0 \quad \ldots \ldots \ldots \quad (3)$$

Nous dirons (comme dans la définition 2 de la première communication) *que S possède la propriété Q si à tout système B correspondent un nombre positif $c = c(B)$ et un indice N_0 indépendant de N, tels que*

$$\frac{H(N, B, S)}{\prod\limits_{\nu=1}^{n} (\beta_\nu - \alpha_\nu)} \geqq c(B), \quad \text{si} \quad N \geqq N_0 . \quad \ldots \ldots \quad (4)$$

Nous dirons que S possède la propriété Q^*, s'il est possible de choisir le nombre c indépendant du système B.

Dans la première communication j'ai donné plusieurs exemples de systèmes S possédant la propriété Q^*. Dans § 2 de la présente communication je démontre le

Théorème 5. *Soient $m_1, m_2, \ldots, m_n$ un système de $n \geqq 1$ nombres naturels arbitraires. Alors le système S des n suites x^{m_ν} $(x = 1, 2, \ldots, 1 \leqq \nu \leqq n)$ possède la propriété Q^*, et donc a fortiori la propriété $\mathfrak{R}$.*

II. Dans § 3 de la première communication, j'ai démontré le théorème 3. La conclusion finale de cette démonstration (le lecteur peut s'en assurer d'un coup d'oeil) entraîne non seulement le théorème 3, mais aussi le théorème suivant dont la rédaction est moins élégante que celle du théorème 3, mais contient quelques détails, dont j'ai besoin ici.

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[1] Première communication, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, p. 176—183 (1942).

Théorème 6. *Soit n un nombre naturel et $\omega_\nu(x)$ pour $\nu = 1, 2, \ldots, n$ une fonction positive non-croissante du nombre naturel x, telle que la série*

$$\sum_{x=1}^{\infty} \prod_{\nu=1}^{n} \omega_\nu(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

diverge et satisfaisant à

$$\omega_\nu(x) \leqq \tfrac{1}{2} \; (\nu = 1, 2, \ldots, n; \; x \geqq 1); \quad x \prod_{\nu=1}^{n} \omega_\nu(x) \rightarrow 0, \quad \text{si} \quad x \rightarrow \infty.$$

Soient S un système de n suites croissantes de nombres naturels (1) *et B un système de $2n$ nombres α_ν, β_ν avec $0 \leqq \alpha_\nu < \beta_\nu \leqq 1$, tels que* (4) *est valable, où $c(B)$ et N_0 désignent des nombres positifs convenablement choisis et où $H(N, B, S)$ est défini par* (2). *Alors la mesure au sens* DE LEBESGUE *de l'ensemble des points $(\theta_1, \ldots, \theta_n)$ du parallélépipède $\alpha_\nu < u_\nu < \beta_\nu$ $(\nu = 1, 2, \ldots, n)$ pour lesquels le système des inégalités*

$$|\theta_\nu f_\nu(x) - y_\nu| < \omega_\nu(x) \quad (\nu = 1, 2, \ldots, n) \quad . \quad . \quad . \quad . \qquad (6)$$

possède une infinité de solutions entières $x \geqq 1$, $y_1, y_2, \ldots, y_n$ est au moins égale à

$$\frac{(c(B))^2}{24 \cdot 2^n \cdot \prod\limits_{\nu=1}^{n} \left(1 + \dfrac{1}{\beta_\nu - \alpha_\nu}\right)} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu).$$

III. Dans la première communication j'ai étudié des systèmes S possédant la propriété Q^* sous la condition supplémentaire $d_\nu(z, x) \rightarrow \infty$, si $x \rightarrow \infty$ $(1 \leqq \nu \leqq n)$ (Théorème 4). Dans la présente communication je considère, sous une condition supplémentaire d'un autre caractère, des systèmes S possédant la propriété $\mathcal{R}$.

Définition 4. *Soient C un nombre positif, $q_1, q_2, \ldots, q_n$ un système de $n \geqq 1$ nombres naturels et S un système de n suites de nombres naturels croissants* (1). *Nous dirons que S possède la propriété $\mathcal{M}(q_1, q_2, \ldots, q_n; C)$, si pour tout nombre $f_\nu(x)$ de la suite* (1) *le nombre $q_\nu f_\nu(x)$ fait partie de cette suite aussi $(1 \leqq \nu \leqq n)$ et si en outre l'indice $\bar{x} = \bar{x}(x; q_1, \ldots, q_n; C)$ pour lequel $q_\nu f_\nu(x) = f_\nu(\bar{x})$ est indépendant de ν et satisfait à l'inégalité $\bar{x} \leqq Cx$* [2].

Exemples. 1. Il est clair que le système S du théorème 5 pour tout nombre naturel q possède la propriété $\mathcal{M}(q^{m_1}, q^{m_2}, \ldots q^{m_n}; q)$.

2. Si $k_1, \ldots, k_n$ désignent n nombres entiers $\geqq 2$, le système des n suites k_ν^x $(x = 1, 2, \ldots)$ possède la propriété $\mathcal{M}(k_1, \ldots, k_n; 2)$.

IV. Dans le § 3 de cette communication je vais démontrer le théorème suivant qui forme une généralisation directe du théorème célèbre de M. A. KHINTCHINE, cité dans la première communication comme Théorème 1*A*, et contient celui là comme cas spécial.

Comme dans la première communication je ferai usage des belles idées développées par M. A. KHINTCHINE dans sa démonstration du théorème 1*A* [3].

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[2] Les nombres $q_1, \ldots, q_n$, C soient indépendants de x.

[3] A. KHINTCHINE, Zur metrischen Theorie der diophantischen Approximationen. Math. Z. **24**, 706—714 (1926).

Théorème 7. *Soient n un nombre naturel et $\sigma_1, \dots, \sigma_n$ un système de n nombres fixes, non négatifs, tels que $\sigma_1 + \dots + \sigma_n = 1$. Soit $\omega_\nu(x)$ pour $\nu = 1, 2, \dots, n$ une fonction positive du nombre naturel x, telle que $x^{\sigma_\nu}\, \omega_\nu(x)$ tend vers zéro monotonement, si $x \to \infty$ et satisfaisant à*

$$\omega_\nu(x) \leqq \tfrac{1}{2} \quad (0 \leqq \nu \leqq n;\ x \geqq 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Soit S un système de n suites de nombres naturels croissants (1) possédant les propriétés $\mathfrak{R}$ et $\mathfrak{M}(q_1, \dots, q_n; C)$ ($q_1, \dots, q_n$ désignant des nombres naturels $\geqq 2$ et C désignant un nombre > 1). Alors pour presque tous les points $(\theta_1, \dots, \theta_n)$ de l'espace R_n le système des inégalités simultanées (7) admet une infinité de solutions entières $x \geqq 1$, $y_1, \dots, y_n$.

§ 2. *Démonstration du théorème 5.*

I. Considérons la suite x^{m_ν} $(x = 1, 2, \dots)$, ν désignant un indice arbitraire $(1 \leqq \nu \leqq n)$. Si α_ν, β_ν désignent des nombres quelconques avec $0 \leqq \alpha_\nu < \beta_\nu \leqq 1$, il est clair que le nombre $A_\nu(x, \alpha_\nu, \beta_\nu)$ est au moins égal au nombre des nombres naturels de l'intervalle $\alpha_\nu f_\nu(x) < u < \beta_\nu f_\nu(x)$ qui sont premiers avec x, c'est à dire que $A_\nu(x, \alpha_\nu, \beta_\nu)$ est au moins

$$(\beta_\nu - \alpha_\nu)\, x^{m_\nu} - \left((\beta_\nu - \alpha_\nu) \frac{x^{m_\nu}}{p_1} + \dots + (\beta_\nu - \alpha_\nu) \frac{x^{m_\nu}}{p_s} \right)$$

$$+ \left((\beta_\nu - \alpha_\nu) \frac{x^{m_\nu}}{p_1 p_2} + \dots + (\beta_\nu - \alpha_\nu) \frac{x^{m_\nu}}{p_{s-1} p_s} \right) -$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\pm (\beta_\nu - \alpha_\nu) \frac{x^{m_\nu}}{p_1 p_2 \dots p_s} - 2^{s+1} =$$

$$= (\beta_\nu - \alpha_\nu)\, x^{m_\nu} \prod_{\sigma=1}^{s} \left(1 - \frac{1}{p_\sigma}\right) - 2^{s+1} = (\beta_\nu - \alpha_\nu)\, \varphi(x^{m_\nu}) - 2^{s+1},$$

si $p_1, p_2, \dots, p_s$ désignent les facteurs premiers différents de x et $\varphi(x)$ désigne la fonction d'EULER. Comme

$$\varphi(x^{m_\nu}) = x^{m_\nu - 1} \varphi(x),$$

nous aurons donc

$$\frac{A_\nu(x, \alpha_\nu, \beta_\nu)}{x^{m_\nu}} \geqq (\beta_\nu - \alpha_\nu) \frac{\varphi(x)}{x} - \frac{2^{s+1}}{x^{m_\nu}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

II. Démontrons maintenant la relation bien connue de MERTENS

$$\sum_{x=1}^{N} \varphi(x) = \frac{3}{\pi^2} N^2 + O(N \log N). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Si $\mu(x)$ désigne la fonction de MOEBIUS, on a

$$\sum_{x=1}^{N} \varphi(x) = \sum_{x=1}^{N} x \sum_{d/x} \frac{\mu(d)}{d} = \sum_{d=1}^{N} \frac{\mu(d)}{d} \sum_{z=1}^{\left[\frac{N}{d}\right]} d \,.\, z = \sum_{d=1}^{N} \mu(d) \left\{ \frac{1}{2} \left[\frac{N}{d}\right] \left(\left[\frac{N}{d}\right] + 1 \right) \right\} =$$

$$= \tfrac{1}{2} N^2 \sum_{d=1}^{N} \frac{\mu(d)}{d^2} + \tfrac{1}{2} N \sum_{d=1}^{N} \frac{\mu(d)}{d} + \sum_{d=1}^{N} O\left(\frac{N}{d}\right) = \tfrac{1}{2} N^2 \,.\, \frac{1}{\zeta(2)} + O(N \log N)\,; \text{ c.q. f.d.}$$

Soit c_1 un nombre positif fixe $< \frac{3}{\pi^2}$. Alors d'après (9) à c_1 un indice N_1 correspond, tel que

$$\sum_{x=1}^{N} \varphi(x) > c_1 N^2, \quad \text{si} \quad N \geqq N_1 . \quad \ldots\ldots \quad (10)$$

Soit posé maintenant $c_2 + c_3 = c_1$, c_2 et c_3 désignant des nombres positifs fixes. Alors j'assers que le nombre des x ($1 \leqq x \leqq N$, N étant $\geqq N_1$) pour lesquels

$$\varphi(x) \geqq c_2 x, \quad \ldots\ldots \quad (11)$$

est au moins égal à

$$c_3 N.$$

En effet, $\varphi(x)$ étant $\leqq N$, on aurait dans le cas contraire

$$\sum_{x=1}^{N} \varphi(x) < c_3 N^2 + \sum_{x=1}^{N} c_2 x \leqq c_3 N^2 + \tfrac{1}{2} c_2 (N^2 + N) \leqq c_1 N^2$$

ce qui nous amène à une contradiction avec (10).

III. Supposons que l'indice x satisfait à (11). Alors nous tirons de (8)

$$\frac{A_\nu(x, \alpha_\nu, \beta_\nu)}{x^{m_\nu}} \geqq c_2 (\beta_\nu - \alpha_\nu) - \frac{2^{s+1}}{x^{m_\nu}} \text{ et donc } > \frac{c_2}{2} (\beta_\nu - \alpha_\nu), \text{ si } x \geqq x_0^{(\nu)},$$

où $x_0^{(\nu)}$ désigne un nombre positif convenablement choisi, ne dépendant que de α_ν et β_ν, car on a

$$s! \leqq p_1 p_2 \ldots p_s \leqq x \text{ et donc } \frac{2^{s+1}}{x^{m_\nu}} \to 0, \text{ quand } x \to \infty.$$

Nous concluons qu'à tout système B de $2n$ nombres α_ν, β_ν ($0 \leqq \alpha_\nu < \beta_\nu \leqq 1$; $\nu = 1, 2, \ldots, n$) un indice x_0, ne dépendant que de B correspond, tel que pour tout $x \geqq x_0$ satisfaisant à (11):

$$\prod_{\nu=1}^{n} \frac{A_\nu(x, \alpha_\nu, \beta_\nu)}{x^{m_\nu}} \geqq \left(\frac{c_2}{2}\right)^n \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu). \quad \ldots\ldots \quad (12)$$

D'après II parmi les nombres $x = 1, 2, \ldots, N$ ($N \geqq N_1$) il y a $c_3 N$ au moins satisfaisant à (11). Si N_0 est convenablement choisi $\geqq x_0$ et $\geqq N_1$ (ne dépendant que de B) parmi les nombres $x = 1, 2, \ldots, N$ ($N \geqq N_0$) il y a donc $\frac{c_3}{2} N$ au moins qui satisfont à (11) et en même temps sont $\geqq x_0$. Pour tous ces x l'inégalité (12) est valable, c'est à dire qu'on a

$$\sum_{x=1}^{N} \prod_{\nu=1}^{n} \frac{A_\nu(x, \alpha_\nu, \beta_\nu)}{x^{m_\nu}} \geqq \frac{c_3 c_2^n}{2^{n+1}} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu).$$

Or, ça veut dire que le système S des n suites x^{m_ν} ($x = 1, 2, \ldots$) possède la propriété Q^*. C.q.f.d.

§ 3. *Démonstration du théorème 7.*

I. J'appliquerai le lemme suivant de M. KHINTCHINE [4]).

Lemme 4. *Si* $\psi(t)$ *désigne une fonction positive et continue de* $t \geqq 1$, *telle que*

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[4]) Cf. [3]).

$t\psi(t)$ tend vers zéro monotonement, si $t \to \infty$ et telle que $\int^{\infty} \psi(t)\,dt$ diverge, une fonction continue $\chi(t)$ existe, telle que

$$\chi(t) \to \infty \text{ monotonement, quand } t \to \infty, \quad \ldots\ldots \quad (13)$$

$$\text{l'intégrale } \int^{\infty} \psi(t\,\chi(t))\,dt \text{ diverge.} \quad \ldots\ldots \quad (14)$$

II. Définissons la fonction continue $\psi_\nu(t)$ par

$$\psi_\nu(x) = \omega_\nu(x) \quad (x \geqq 1, \text{ entier}) \quad t^{\sigma_\nu}\psi_\nu(t) = \text{linéaire} \quad (x \leqq t < x+1) \quad (15)$$

et posons

$$\psi(t) = \prod_{\nu=1}^{n} \psi_\nu(t).$$

Alors le lemme 4 nous apprend l'existence d'une fonction positive et continue $\chi(t)$ possédant les propriétés (13) et (14). D'après une proposition bien connue, on peut choisir une fonction positive, continue $\eta(t) \leq 1$, tendant vers zéro monotonement, si $t \to \infty$, telle que l'intégrale

$$\int^{\infty} \eta(t)\,\psi(t\,\chi(t))\,dt$$

diverge encore. En remarquant que la fonction $(t\chi(t))^{\sigma_\nu}\psi_\nu(t\chi(t))$ tend vers zéro monotonement, si $t \to \infty$ et en posant

$$\Omega_\nu(t) = \sqrt[n]{\eta(t)\,\psi_\nu(t\,\chi(t))} \quad (1 \leqq \nu \leqq n), \quad \ldots\ldots \quad (16)$$

je conclue que les fonctions $t^{\sigma_\nu}\Omega_\nu(t)$ et $\Omega_\nu(t)$ tendent vers zéro monotonement, quand $t \to \infty$ et que l'intégrale

$$\int^{\infty} \prod_{\nu=1}^{n} \Omega_\nu(t)\,dt$$

et donc aussi la série

$$\sum_{x=1}^{\infty} \prod_{\nu=1}^{n} \Omega_\nu(x)$$

divergent.

Parce qu'on a en outre $\Omega_\nu(t) \leq \frac{1}{2}$, il est clair que les conditions du théorème 6 sont remplies, si l'on y pose $\alpha_\nu = 0$, $\beta_\nu = 1$ (*c'est à dire* $B = B_0$) et $\Omega_\nu(x)$ au lieu de $\omega_\nu(x)$ $(\nu = 1, 2, \ldots, n)$ et si S y désigne le système du théorème 7. Nous en concluons que la mesure mG de l'ensemble G des points $(\theta_1, \ldots, \theta_n)$ du cube $0 < u_\nu < 1$ $(\nu = 1, 2, \ldots, n)$ pour lesquels le système

$$|\theta_\nu f_\nu(x) - y_\nu| < \Omega_\nu(x) \quad (\nu = 1, 2, \ldots, n) \quad \ldots\ldots \quad (17)$$

admet une infinité de solutions entières $x \geqq 1, y_1, \ldots, y_n$ est positive.

III. Soit r un nombre naturel et partageons le cube $0 \leqq u_\nu \leqq 1$ en $q_1^r\, q_2^r \ldots q_n^r$ parallélépipèdes P, congruents entre eux, dont les cotés (ayant les longueurs $q_1^{-r}, \ldots, q_n^{-r}$) sont parallèles à ceux du cube. Soit ε un nombre positif et < 1. Alors $r = r(\varepsilon)$ étant choisi suffisamment grand, dans au moins un des parallélépipèdes P (désignons le par P_0), la densité de G est au moins égal à $1-\varepsilon$. À chacun des P une translation congruente

$$\theta_\nu = \theta'_\nu + \frac{h_\nu}{q_\nu^r} \qquad (h_\nu \text{ entier}; \quad \nu = 1, 2, \ldots, n) \ . \ . \ . \quad (18)$$

correspond, transférant les points $(\theta'_1, \ldots, \theta'_n)$ de P_0 en des points $(\theta_1, \ldots, \theta_n)$ de P. Soit $(\theta_1, \ldots, \theta_n)$ un point de P_0 appartenant à G et soit $x \geqq 1, y_1, \ldots, y_n$ une solution entière quelconque de (17). Alors pour tout point $(\theta'_1, \ldots, \theta'_n)$ avec (18) on a

$$\left.\begin{array}{r} |\theta'_\nu q_\nu^r f_\nu(x) - q_\nu^r y_\nu + h_\nu f_\nu(x)| = |\theta_\nu f_\nu(x) - y_\nu| q_\nu^r < q_\nu^r \Omega_\nu(x) \\ (\nu = 1, 2, \ldots, n) \end{array}\right\}. \quad (19)$$

Le système S possédant la propriété $\mathfrak{M}(q_1, \ldots, q_n; C)$ nous pouvons poser

$$q_\nu^r f_\nu(x) = f_\nu(X), \quad \text{où} \quad X \leqq C^r x.$$

En posant en outre $Y_\nu = q_\nu^r y_\nu - h_\nu f_\nu(x)$ nous tirons de (19)

$$|\theta'_\nu f_\nu(X) - Y_\nu| < q_\nu^r \Omega_\nu\left(\frac{X}{C^r}\right)$$

et donc, si x est suffisamment grand

$$< \psi_\nu\left(\frac{X}{C^r}\chi\left(\frac{X}{C^r}\right)\right) < \psi_\nu(X) = \omega_\nu(X) \qquad (\nu = 1, 2, \ldots, n),$$

à cause des relations (16), $\eta(x) \to 0$, $\chi(x) \to \infty$ (quand $x \to \infty$ et (15).

Comme (17) possède une infinité de solutions entières nous concluons que le système

$$|\theta'_\nu f_\nu(X) - Y_\nu| < \omega_\nu(X) \qquad (\nu = 1, 2, \ldots, n)$$

admet une infinité de solutions entières de même. Ça veut dire que le point $(\theta'_1, \ldots, \theta'_n)$ appartient à G. Or, ceci entraîne que la densité de G par rapport à chacun des parallélépipèdes P est $\geqq 1-\varepsilon$. Le nombre ε étant arbitraire et G ne dépendant pas de ε, nous en concluons que la mesure mG est égale à 1. C.q.f.d.

Hydrodynamics. — *Laminar flow in radial direction along a plane surface.* By A. VAN WIJNGAARDEN. (Mededeeling N⁰. 43 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of February 28, 1942.)

1. The theory of laminar boundary layer flow mostly has been used for those types of flow in which the velocity of the fluid at great distances from the wall is approximately parallel to a single given direction. A solution of this kind for the case of the flow in radial direction along an infinite plane surface has been given by HOMANN[1]. In this solution at great distances from the plane the velocity is directed towards the plane; however, at the same time it possesses a radial component which increases proportionally to the distance from the central axis. The equation of continuity then requires that the component normal to the plane shall increase indefinitely with the distance from the plane; it must be assumed, consequently, that the field is limited by another plane at some great distance. In this case the amount of fluid transported in the radial direction through the surface of a cylinder having its axis along the axis of the field increases proportionally to the square of the radius of the cylinder, in agreement with the fact that the amount of fluid supplied by the main stream likewise is proportional to the square of the radius.

In the case where a limited amount of fluid is supplied in the axis of the field, the velocity component in the radial direction must decrease inversely proportionally to the radius. We meet such kinds of flow for instance in the valves of piston pumps and in the air supported bearings of modern ultra-centrifuges. Actually the problem here is greatly complicated by the presence of a second limiting plane boundary.

The investigation of the field with decreasing radial velocity along a single plane wall will be the subject of the present paper. In a final remark a more general view on the problem has been indicated.

2. We take the plane boundary as X, Y-plane and the central axis as Z-axis. Further we denote the distance from the axis by r, and the velocities in the r- and z-directions by u and w respectively. It is supposed that the motion is independent of the time and has no tangential component. Then the equations of motion and of continuity are:

$$\left.\begin{aligned} & u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{\partial}{\partial r}\left(\frac{p}{s}\right)+\nu\left[\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right] \\ & u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial}{\partial z}\left(\frac{p}{s}\right)+\nu\left[\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right] \\ & u=\frac{1}{r}\frac{d\psi}{\partial z}; \qquad w=-\frac{1}{r}\frac{\partial\psi}{\partial r} \end{aligned}\right\} . \tag{1}$$

Here p is the pressure, s the (constant) density and ν the kinematic viscosity of the fluid. By introducing the dimensionless variables:

$$\varrho=\frac{r}{r_0};\ \zeta=\frac{z}{r_0\sqrt{\varepsilon}};\ u^*=\frac{u}{u_0};\ w^*=\frac{w}{u_0\sqrt{\varepsilon}};\ \left(\frac{p}{s}\right)^*=\left(\frac{p}{s}\right)\Big/u_0{}^2;\ \psi^*=\frac{\psi}{r_0\sqrt{\varepsilon}},$$

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[1] F. HOMANN, Zeitschr. f. angew. Math. u. Mech. **16**, 153—164 (1936).

where r_0 and u_0 are certain convenient standard measures of length and velocity and ε is the inverse of the Reynolds number $R = u_0 r_0/\nu$, the equations (1) change into:

$$\left.\begin{aligned}
&u^* \frac{\partial u^*}{\partial \varrho} + w^* \frac{\partial u^*}{\partial \zeta} = -\frac{\partial}{\partial \varrho}\left(\frac{p}{s}\right)^* + \frac{\partial^2 u^*}{\partial \zeta^2} + \varepsilon\left[\frac{\partial^2 u^*}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial u^*}{\partial \varrho} - \frac{u^*}{\varrho^2}\right]\\
&\varepsilon\left[u^* \frac{\partial w^*}{\partial \varrho} + w^* \frac{\partial w^*}{\partial \zeta}\right] = -\frac{\partial}{\partial \zeta}\left(\frac{p}{s}\right)^* + \varepsilon \frac{\partial^2 w^*}{\partial \zeta^2} + \varepsilon^2\left[\frac{\partial^2 w^*}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial w^*}{\partial \varrho}\right]\\
&u^* = \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \zeta};\; w^* = -\frac{1}{\varrho}\frac{\partial \psi^*}{\partial \varrho}
\end{aligned}\right\} . \quad (1a)$$

By crossdifferentiating these equations we eliminate the pressure; the resulting equation for ψ^* becomes:

$$\left.\begin{aligned}
&\frac{\partial^4 \psi^*}{\partial \zeta^4} + \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial^3 \psi^*}{\partial \zeta^3} + \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \zeta}\left(\frac{2}{\varrho}\frac{\partial^2 \psi^*}{\partial \zeta^2} - \frac{\partial^3 \psi^*}{\partial \varrho\, \partial \zeta^2}\right) =\\
&= \varepsilon\left[-2\frac{\partial^4 \psi^*}{\partial \varrho^2\, \partial \zeta^2} + \frac{2}{\varrho}\frac{\partial^3 \psi^*}{\partial \varrho\, \partial \zeta^2} + \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \zeta}\frac{\partial^3 \psi^*}{\partial \varrho^3} - \frac{1}{\varrho}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial^3 \psi^*}{\partial \varrho^2\, \partial \zeta} -\right.\\
&\qquad \left. - \frac{3}{\varrho^2}\frac{\partial \psi^*}{\partial \zeta}\frac{\partial^2 \psi^*}{\partial \varrho^2} + \frac{1}{\varrho^2}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial^2 \psi^*}{\partial \varrho\, \partial \zeta} + \frac{3}{\varrho^3}\frac{\partial \psi^*}{\partial \varrho}\frac{\partial \psi^*}{\partial \zeta}\right] +\\
&\qquad + \varepsilon^2\left[-\frac{\partial^4 \psi^*}{\partial \varrho^4} + \frac{2}{\varrho}\frac{\partial^3 \psi^*}{\partial \varrho^3} - \frac{3}{\varrho^2}\frac{\partial^2 \psi^*}{\partial \varrho^2} + \frac{3}{\varrho^3}\frac{\partial \psi^*}{\partial \varrho}\right]
\end{aligned}\right\} . \quad (2)$$

Now we bound the field by a cone having its axis along the Z-axis and its top in the origin. A generating line of the cone is given by: $z = M \, . \, r$. If we write $\eta = \zeta/\varrho$, this generating line is given by $\eta = M/\sqrt{\varepsilon} = N$.

The boundary conditions for $\eta = 0$ always are: $u^* = 0$ and $w^* = 0$. Along with these we give the velocity at the surface of the cone $\eta = N$ by prescribing here: $u^* = 1/\varrho$ and $w^* = C/\varrho$. In this case a solution may be found, assuming:

$$\psi^* = \varrho \, . \, f(\eta). \quad \ldots\ldots\ldots \quad (3)$$

For f we obtain the equation:

$$\left.\begin{aligned}
&f^{\mathrm{IV}} + f''' f + 3 f'' f' =\\
&= \varepsilon[-2f'' - 6\eta f''' - 2\eta^2 f^{\mathrm{IV}} + 3ff' - 3\eta f'^2 - 3\eta^2 f' f'' - 3\eta ff'' - \eta^2 ff'''] +\\
&\qquad + \varepsilon^2[3f - 3\eta f' - 21\eta^2 f'' - 10\eta^3 f''' - \eta^4 f^{\mathrm{IV}}]
\end{aligned}\right\} . \quad (4)$$

As $u^* = f'/\varrho$ and $w^* = (\eta f' - f)/\varrho$ the boundary conditions become:

$$\text{for } \eta = 0 : f = 0, f' = 0$$
$$\text{for } \eta = N : f = \eta - C, f' = 1.$$

3. Equation (4) can be integrated once and then becomes:

$$\left.\begin{aligned}
&f''' + f'^2 + ff'' + A_0 =\\
&= \varepsilon[-2\eta f'' - 2\eta^2 f''' + 2f^2 - \eta^2 f'^2 - \eta ff' - \eta^2 ff''] +\\
&\qquad + \varepsilon^2[3\eta f - 3\eta^2 f' - 6\eta^3 f'' - \eta^4 f''']
\end{aligned}\right\} . \quad . \quad (5)$$

In order to obtain an impression of the order of magnitude of the terms on the right hand side we substitute the asymptotic expansion:

$$f = \eta - C + a_1/\eta + a_2/\eta^2 + \dots$$

The terms between the square brackets then become, respectively:

$$-3\eta C + (2C^2 + 4a_1) + 0(1/\eta) \text{ and } -3\eta C + 0(1/\eta).$$

As the maximum value of η is given by $M/\sqrt{\varepsilon}$ and as we will consider such solutions only in which f and its derivatives remain finite for all η below this maximum value, the order of magnitude of the right hand side of equation (5) at most will be: $M\sqrt{\varepsilon}$. Consequently for every finite value of M we can make the right hand side arbitrarily small by choosing the Reynolds number R sufficiently high. For great values of R we therefore may confine ourselves to the solution of the equation:

$$f''' + f''f + f'^2 - A_0 = 0,$$

with the boundary conditions $f = 0$, $f' = 0$ for $\eta = 0$; and $f = \eta - C$, $f' = 1$ for $\eta = \infty$. The last condition gives $A_0 = 1$. Hence:

$$f''' + f''f + f'^2 - 1 = 0 \quad \dots\dots\dots \quad (6)$$

This is a boundary layer equation which might also have been deduced in the ordinary way by taking for the pressure the "undisturbed" pressure and dropping all terms of the order of ε. The method used above, however, has the advantage of admitting a precise check upon the possible influence of the neglected terms. The same as nearly all boundary layer equations, equation (6) is a special case of HARTREE's equation [2]:

$$f''' + f''f - \beta(f'^2 - 1) = 0,$$

with $\beta = -1$. From HARTREE's analysis we know that, as our β has a negative value, the ordinary condition $f' = 1$ for $\eta = \infty$ is not sufficient to determine the function f, and therefore we are able to apply the more precise condition $f = \eta - C$.

In our special case the equation lends itself to a more rigorous investigation. The equation can be integrated twice and then becomes:

$$f' + \tfrac{1}{2}f^2 - \tfrac{1}{2}\eta^2 + B\eta + D = 0.$$

The boundary conditions give $D = 0$ and $B = C$; hence we obtain:

$$f' + \tfrac{1}{2}f^2 - \tfrac{1}{2}\eta^2 + C\eta = 0. \quad \dots\dots\dots \quad (7)$$

As $C = -f''(0)$ we can integrate this equation numerically, starting with a given value of C. It only has to be demonstrated that f will tend to $\eta - C$ with increasing values of η for every value of C.

If $C \neq 0$ we can make the substitutions: $\eta = Cz$, $f = CF$; the equation then is transformed into:

$$dF/dz = C^2(\tfrac{1}{2}z^2 - z - \tfrac{1}{2}F^2). \quad \dots\dots\dots \quad (8)$$

This form lends itself very well for a graphical discussion, as dF/dz has a known value on every hyperbola

$$\tfrac{1}{2}z^2 - z - \tfrac{1}{2}F^2 = A = \textit{constant}.$$

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[2] D. R. HARTREE, Proc. Camb. Phil. Soc. **33**, 223—239 (1937).

It is easily shown (see fig. 1) that for all negative values, as well as for all positive values of C up to a certain limit, the limiting form of F indeed will be $z - 1$ as required. There is a critical value of C for which the limiting form becomes, however, $-z + 1$.

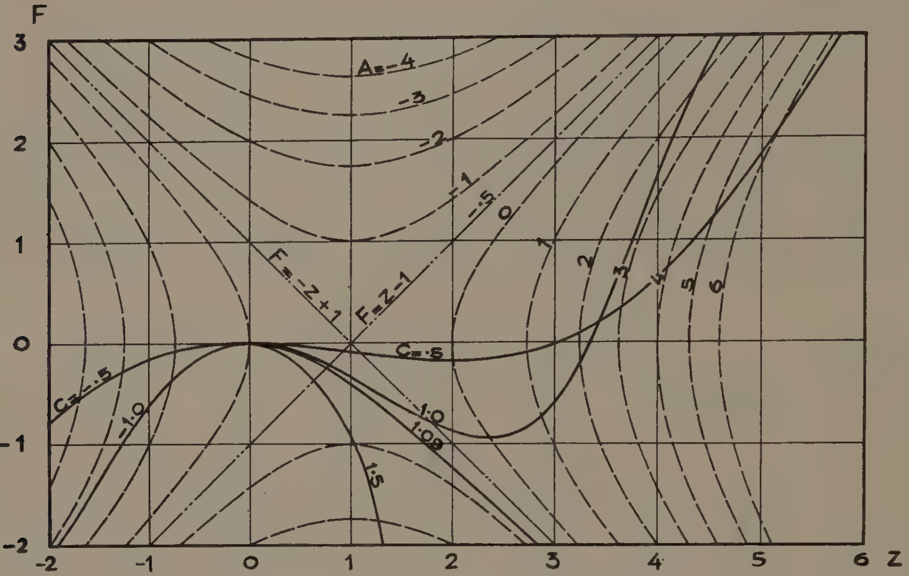

Fig. 1. Graphical solution of equation (8).

For greater values of C the solutions again will have the limiting form $z - 1$, but now they have a pole for some finite value of z. Then comes a second critical value of C, for which the limiting form again is: $-z - 1$; after that the solution has two poles and so on. The first critical value of C appears to be about 1,09.

4. Analytically the behaviour of f can be discussed by substituting $f = 2h'/h$, $x = \eta - C$ and $\frac{1}{4}C^2 = n + \frac{1}{2}$. Then equation (7) changes into:

$$d^2h/dx^2 + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)\,h = 0 \quad . \; . \; . \; . \; . \; . \; . \qquad (9)$$

This is WEBER's equation defining the parabolic cylinder functions[3]). We can choose two fundamental solutions $D_n(x)$ and $E_n(x)$ with the asymptotic expansions:

$$D_n(x) \backsim e^{-x^2/4} \left\{ x^n - \frac{n(n-1)}{2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4} x^{n-4} - \dots \right\}$$

$$E_n(x) \backsim e^{x^2/4} \left\{ x^{-n-1} + \frac{(n+1)(n+2)}{2} x^{-n-3} + \right.$$

$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4} x^{-n-5} + \dots \right\}.$$

Then:

$$f \backsim \frac{-e^{-x^2/4} \, . \, x^{n+1} + a\, e^{x^2/4} \, . \, x^{-n}}{e^{-x^2/4} \, . \, x^n + a\, e^{x^2/4} \, . \, x^{-n-1}} \quad . \; . \; . \; . \; . \; . \; . \qquad (10)$$

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[3]) E. T. WHITTAKER and G. N. WATSON, Modern Analysis, (Cambridge, 1920), p. 347, § 16. 5.

α has to be determined in such a way that the condition $f = 0$ for $\eta = 0$ (*i.e.* for $x = -C$) will be satisfied. This requires:

$$\alpha = -D_n'(-C)/E_n'(-C). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now if $\alpha \neq 0$, we have $f \sim x$ for great values of x; however, if $\alpha = 0$, we shall have $\sim -x$. By considering the particular cases where n is an integer it can be shown that there exists one critical value for negative n, and further one critical value between every two consecutive positive integer values of n; the number of poles of a solution is equal to the number of critical values inferior to n.

Hence we see that for every value of C up to about 1,09 we have found a solution that satisfies all boundary conditions. For higher values of C the function f and its derivatives will become infinite for one or more values of η and the neglections introduced into equation (4) no longer are valid.

The numerical computation of f may be executed best by means of a numerical integration of (7), in combination with an easily constructed series in ascending powers of η, applicable for very small η, and with an asymptotic series applicable for very large η. Both series, however, are not easily manageable.

A simple table of values of f, f' and $f - \eta f'$ has been computed for the case $C = 0$. Although for that value of C the function f will approach to its limiting form as quickly

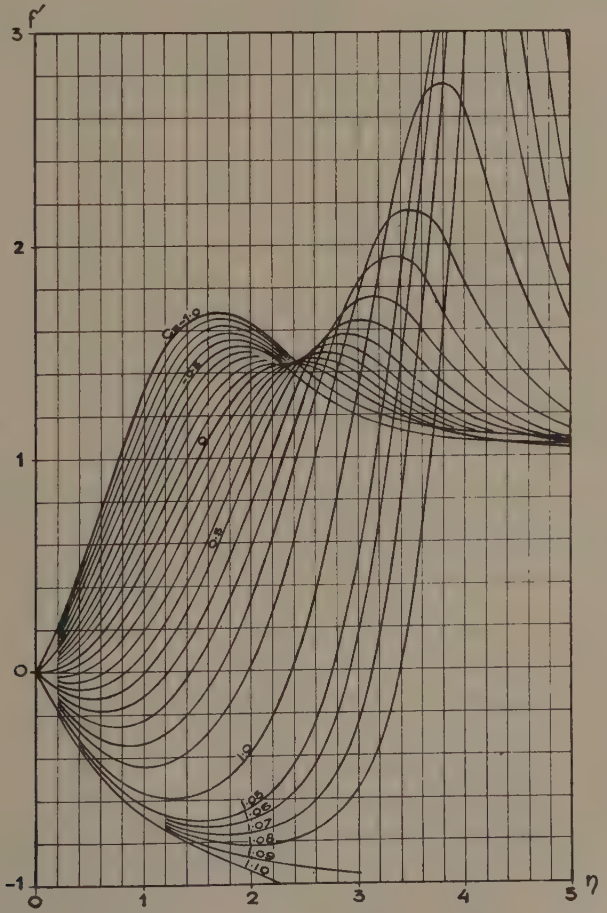


Fig. 2. Solutions of equation (7) for different values of C.

Solution of equation (7) for the case $C = 0$.

η	f	f'	$\eta f'-f$	η	f	f'	$\eta f'-f$
0	0	0	0	4.3	4.041	1.080	0.603
0.1	0.00017	0.00500	0.00043	4.4	4.149	1.074	0.577
0.2	0.00133	0.02000	0.00267	4.5	4.256	1.069	0.555
0.3	0.00450	0.04499	0.00900	4.6	4.363	1.064	0.534
0.4	0.01066	0.07994	0.02131	4.7	4.469	1.060	0.514
0.5	0.02082	0.12478	0.04157	4.8	4.575	1.057	0.498
0.6	0.03594	0.17935	0.07167	4.9	4.680	1.054	0.482
0.7	0.05700	0.24338	0.11336	5.0	4.785	1.050	0.468
0.8	0.08492	0.31639	0.16819	5.5	5.308	1.039	0.410
0.9	0.12056	0.39773	0.23740	6.0	5.826	1.032	0.367
1.0	0.16471	0.48644	0.32173	6.5	6.340	1.027	0.334
1.1	0.21805	0.58123	0.42130	7.0	6.853	1.022	0.305
1.2	0.28111	0.68049	0.53548	7.5	7.363	1.019	0.283
1.3	0.35423	0.78226	0.66271	8.0	7.872	1.017	0.263
1.4	0.43757	0.88427	0.80041	8.5	8.380	1.015	0.246
1.5	0.53101	0.98401	0.94500	9.0	8.887	1.013	0.231
1.6	0.63421	1.07889	1.09201	9.5	9.393	1.012	0.218
1.7	0.74654	1.16634	1.23624	10	9.898	1.011	0.206
1.8	0.86715	1.24403	1.37210	11	10.908	1.008	0.187
1.9	0.99495	1.31003	1.49411	12	11.916	1.007	0.170
2.0	1.12872	1.36299	1.59726	13	12.922	1.006	0.157
2.1	1.26710	1.40223	1.67758	14	13.928	1.005	0.145
2.2	1.40871	1.42777	1.73238	15	14.933	1.004	0.135
2.3	1.55222	1.44031	1.76049	16	15.937	1.004	0.126
2.4	1.69638	1.44114	1.76236	17	16.941	1.003	0.119
2.5	1.84012	1.43200	1.73988	18	17.944	1.003	0.112
2.6	1.9825	1.4148	1.6960	19	18.947	1.003	0.105
2.7	2.1229	1.3917	1.6347	20	19.950	1.002	0.101
2.8	2.2607	1.3646	1.5602	25	24.960	1.002	0.080
2.9	2.3957	1.3353	1.4767	30	29.967	1.001	0.067
3.0	2.5277	1.3053	1.3882	35	34.971	1.001	0.057
3.1	2.6568	1.2757	1.2979	40	39.975	1.001	0.050
3.2	2.7829	1.2476	1.2094	50	49.980	1.000	0.040
3.3	2.9064	1.2216	1.1249	60	59.983	1.000	0.033
3.4	3.0273	1.1978	1.0452	70	69.986	1.000	0.028
3.5	3.1460	1.1763	0.9711	80	79.988	1.000	0.025
3.6	3.2626	1.1576	0.9048	90	89.989	1.000	0.022
3.7	3.3775	1.1412	0.8449	100	99.990	1.000	0.020
3.8	3.4909	1.1268	0.7909	200	199.995	1.000	0.010
3.9	3.6029	1.1146	0.7440	300	299.997	1.000	0.007
4.0	3.7138	1.1037	0.7010	400	399.998	1.000	0.005
4.1	3.824	1.095	0.666	500	499.998	1.000	0.004
4.2	3.933	1.087	0.632	1000	999.999	1.000	0.001

as possible, the approach still is very slow. For other values of C between —1,0 and + 1,10 the equation has been solved graphically; the results are indicated in fig. 2.

5. *Final remark.* Instead of making the particular assumption that for large values of z the velocity u^* should decrease inversely proportionally to ϱ, we also may try to find a solution for the case in which u^* is proportional to an arbitrary power of ϱ, by assuming $\psi^* = \varrho^{\alpha} . f(\eta)$, where $\eta = \zeta . \varrho^{-\gamma}$. As the powers of ϱ on the left hand side of equation (2) at any rate must be the same, we obtain the condition:

$$\alpha + \gamma = 2.$$

Now the terms on the right hand side of the equation generally are not of the same order in ϱ as the terms on the left hand side. For $\alpha = 1$ (which is the case just treated), however, this condition is satisfied. For $\alpha = 2$, which is the case of HOMANN's solution, the right hand side of the equation wholly vanishes. If we bound the field again by a surface of revolution, a meridian curve of which is defined by $\eta = M/\sqrt{\varepsilon}$, then we can ask for what values of α the terms on the right hand side may be made arbitrarily small by choosing the Reynolds number high enough.

The expression between the first pair of brackets on the right hand side in (5) must not have a term with η^2 in its asymptotic expansion. We find that this is the case only for $\alpha = 1$, $\alpha = 2$ or $\alpha = -{}^1/_2$. For other values of α we may neglect the terms on the right hand side only for not too large η. The simplified equation then becomes:

$$f^{\mathrm{IV}} + \alpha f''' f + (6 - 3\alpha) f'' f' = 0,$$

which can be integrated into:

$$f''' + \alpha f'' f + (3 - 2\alpha)(f'^2 - 1) = 0.$$

If we put,

for positive $\alpha : F = f\sqrt{\alpha}\,;\, y = \eta\sqrt{\alpha}\,;$

for negative $\alpha : F = -f\sqrt{-\alpha}\,;\, y = \eta\sqrt{-\alpha}\,;$

and in both cases: $\beta = -(3 - 2\alpha)/\alpha$, the equation becomes:

$$\frac{d^3 F}{dy^3} + F\frac{d^2 F}{dy^2} - \beta(f'^2 - 1) = 0.$$

Hence we obtain HARTREE's equation in its general form. For $\alpha = 1$, $\alpha = 2$ and $\alpha = -{}^1/_2$ the values of β are —1, +½ and +8 respectively. The last two values of β are positive and the solution in this case is determined by the conditions $f = 0$ and $f' = 0$ for $\eta = 0$, and $f' = 1$ for $\eta = \infty$.

Botany. — *The lichenisation of aerophilic algae.* By A. QUISPEL. (From the Botanical Institute, University of Leyden and the Laboratory of microbiology, Delft Technical Institute). (Communicated by Prof. L. G. M. BAAS BECKING.)

(Communicated at the meeting of February 28, 1942.)

Though the dual nature of lichens is generally recognized since the days of SCHWENDENER, there does not exist an equally great unanimity as to the way in which we have to look upon the mutual relations between the two components. The reason for this is, in the first place, the lack of physiological experiments. Physiology was a long time one of the most neglected chapters of lichenology and only in recent times a greater interest seems to have been taken in the physiological problems of lichens. For a better understanding of the symbiosis, however, it does not suffice to experiment with the lichen as a whole, but we have to work with the two isolated components: alga and fungus. Several authors have investigated the algae in pure culture, yet there are still many problems to solve. As to the fungal part, however, the experiments are very scanty. Though it appeared to be possible to cultivate this symbiont, as shown by the work of MOELLER (1887), WERNER (1927) and THOMAS (1939), the growth-velocity in vitro is so small that physiological work needs remained limited to preliminary or simple experiments. Moreover, as the fungi grow badly in liquids, quantitative work is impeded.

We believe to have found in the lichenized algal covers of Pleurococcus, Apatococcus and allied species, a better object of study in this respect. The observation that these algal covers are always mixed with symbiontic hyphae is due to SCHMID (1933); his endeavours remained limited to slide-cultures. During the running of our experiments the comprehensive work of THOMAS (1939) on the biology of lichen-components appeared, in which he describes an attempted isolation of these fungi as well. Though he does not say very much upon this subject, he mentions that growth was better than in the known lichen-fungi and that he had been able to isolate eight different species.

We isolated the hyphae from these algal-covers by the aid of hanging-drop cultures of a suspension in sterile tap-water, in which the hyphae developed, after which the drop was transferred to a tube with malt-agar, or by isolating a small group of algal cells with adhering hyphae with a small glas-capillary under the microscope. In this way several fungi were isolated, which resembled each other in many respects, but differed in details. The lack of typical fructifications did not enable us to make any definite classification. As an illustration the description of one of them is given below:

> Solid, compact, very hard thallus, with a dark greyish-black colour in the hyphae, which colour diffuses into the surrounding agar. Thallus elevated and lobated, on some media for the main part in the agar, with a height equal to its length. Thallus consisting of thin hyphae penetrating into the substrate, a central layer (consisting of interlaced hyphae with thick-walled, more or less rounded cells), an outer layer, in which the cells are long-drawn (10—15 μ long, 4,5—6 μ thick), with thick walls and delicate terminal branches rising into the air. Finally we want to mention the occurrence of large, globate cells and of oil-drops in old cells.

The other fungi differ from the described one in the intensity and nature of the pigment production, the cell-form in the central layer, the dimensions of the cells, the length of the aerial hyphae, etc. In one of the fungi these aerial hyphae consist of small ovoid cells, which loosen easily on suitable media, thus functioning as conidia.

Two points are of a special interest:

1. the characteristics described above are very similar to those described for certain true lichen fungi in literature. Here too we find mentioned the compact hard colonies, the elevated mode of growth, the darkening of the surrounding agar, the stratified thallus, the thick-walled cells, the occurrence of big globular cells (sometimes misinterpreted as algae), the oil production in the protoplasm-rich cells, the conidia production of the aerial hyphae, while many of the pictures given resemble my fungi in every respect. Moreover I isolated a fungus from a heavy lichenized, soredial cover of Cystococcus spec. as they are often to be found in the vicinity of towns. This fungus, which certainly is a lichen fungus (though soredial covers like these cannot be classified), could hardly be distinguished from some of the Pleurococcus and Apatococcus symbionts. The only lichen fungus which I isolated from a lichen (Xanthoria parietina (L) Th. Fr.) does not seem to be identical with one of them, although again there are many points of similarity. Though I cannot establish whether my fungi are identical with certain true lichen fungi, a close relationship with at least some of them seems to be beyond any doubt.

2. Yet the growth velocity, even of the fungus from the soredial Cystococcus cover, is much better. After inoculating a malt-agar plate with a suspension of hyphae, we obtain after two weeks already colonies with a diameter of 4—8 mm. Moreover they grow excellently in liquid culture solutions. In suitable liquid media they form (in ERLENMEYER flasks of 300 cc, provided with 100 cc liquid) mycelia with a dry weight of 1—1,5 gr. after a two month incubation. Most probably our fungi are to be regarded as relatives (or perhaps even strains) of true lichen fungi, which are less adapted to the symbiosis and in consequence growing better in pure culture. An investigation after the physiological properties seemed to be of a great importance for the problem of the lichen symbiosis, though of course we have to be very cautious to apply the results obtained to true lichens, as here conditions will be more complicated than in our more primitive (or reduced?) alga-fungus symbiosis.

Though the investigation after these properties still is in full progress we want to mention here the most important facts, which have come to light.

Whilst the fungi are developing well in media like maltextract, the development on synthetic media is very scanty. The application of some accessory substances in the form of yeast extract seemed to be necessary. Usually this beneficial effect appeared to be caused by aneurin.

TABLE I.

Dry weight of the mycelia of the fungi Pl I, isolated from a Pleurococcus cover and C, isolated from a soredial Cystococcus cover, cultivated on CZAPEK–DOX solution (25 cc in ERLENMEYER flasks of 50 cc with different concentrations of aneurin (MERCK).

Aneurin concentration	0	5	10	15	γ p. 25 cc
Dry weight Pl I	20	310	145	100	mg
Dry weight C	38	90	105	100	mg

The growth requirements of some other fungi appear to be more complicated, as here the beneficial effect of aneurin does not seem as pronounced as that of yeast extract.

It seemed indicated to investigate whether the algal partner could provide this vitamin in nature. A strain of Apatococcus minor Edl. from the collection of the laboratory for microbiology at Delft and three cultures of lichen gonidia (from Xanthoria parietina (L) TH. FR. Physcia pulverulenta (HOFFM.) NYL. and Parmelia acetabulum (NECK) DUB., which were isolated according to the method of JAAG (1929) and cultivated on BEIJERINCK agar with 2 % glucose, appeared to develop well on aneurin-free media, so that the supposition seemed justified that these algae were able to synthesize this vitamin. The convincing proof was obtained by the following experiment:

ERLENMEYER flasks of 50 cc provided with 25 cc CZAPEK–DOX solution without aneurin were inoculated with the algae mentioned above and cultivated in the light. After a month small green globules were clearly visible all over the bottom of the flasks. Then they were inoculated with the same amount of a suspension of the fungus Pl. I, together with two sterile CZAPEK–DOX media, one with and one without aneurin as a control.

TABLE II.

Control	Apat. minor	Xanthor. par.	Parm. acet.	10 γ % aneurin
25	85	90	330	440 mg dry weight

These figures leave no doubt that the relatively small amount of algae present in the solution already had a marked influence. So it is evident, that the algal symbiont provides its fungus partner with the required aneurin. This is the first observation which makes it probable that in the lichen symbiosis, like in so many cases of symbiosis, the exchange of accessory growth substances plays an important role.

As a source of carbon the fungi can make use of different sugars, starch and polyalcohols. Among the last mentioned the fitness of erythritol, one of the most prominent reserve-substances of the proto-pleurococcoid algae and of certain lichen-gonidia is of a special interest. As a matter of course there are specific differences between the fungi in this respect.

Organic and inorganic compounds may be used as nitrogen sources. On nitrogen-free media the fungi show no development, even after the addition of traces of sodium molybdate. In media with small quantities of yeast extract as only source of nitrogen the development, however, was so abundant that it warranted the determination of the nitrogen content as compared with uninoculated control media by an ordinary KJELDAHL method.

TABLE III.

Dry weight and nitrogen content of the fungi Pl 1, Pl 2 and C after cultivation in ERLENMEYER flasks of 300 cc provided with 100 cc nitrogen-free CZAPEK–DOX solution with 1 cc yeast extract after two months incubation.

Fungus	Pl I	Pl 1	Pl 2	C
Dry weight	860	1240	860	510 mg
Nitrogen content (mycelium + medium)	7.39	12.25	12.42	11.95 „
Nitrogen content uninoculated medium	8.05	12.59	12.83	11.99 „

So any assimilation of atmospheric nitrogen is altogether out of the question. On the contrary, we observe in all cases a disappearance of nitrogen as compared to the nitrogen in the uninoculated control, which may be easily explained by the evaporation of some volatile nitrogen compounds during the two months of cultivation. The fungi appear to develop well in solutions relatively poor in nitrogen.

An extensive search was made after the metabolic products of these fungi in relation to an eventual production of lichenic acids. These remarkable, taxonomically important products, which are produced by most lichens in often very considerable amounts were formerly regarded as specific products of the symbiosis. The most convincing arguments for this assumption were that they had been never found in any other organism and especially the old experiment of TOBLER (1909), in which this investigator showed that the lichenic acid parietin (physcion) was only produced by the fungus in pure culture after synthesis with his algal partner.

More recently, however, RAISTRICK c.s. isolated from some very common moulds like Aspergillus and Penicillium a great number of remarkable metabolic products, which in many cases bore a striking resemblance to certain lichenic acids, while finally he could identify one of the metabolic products of Aspergillus glaucus Link. with the lichenic acid parietin. Whilst these observations made it very probable that many of the lichenic acids were the result of the metabolism of the fungus alone, the decisive proof was only very recently given by THOMAS (1939), who was able to demonstrate the existence of parietin in pure cultures of the fungus components of Caloplaca and Xanthoria species and stictaurin in the fungus partner of Candelariella vitellina (EHRH.) MULL. ARG. Whether all lichenic acids are the product of the fungus alone still remains doubtful, in consequence of the chemical diversity of these compounds (see compilation of ASAHINA 1939).

It seemed interesting to investigate whether we could find among the metabolic products of our fungi, which are to be regarded as relatives of the lichen fungi and from which we can cultivate in a relatively short time such great quantities, lichenic acids or allied substances. To this purpose they were cultivated on the most divergent ways: in media with varying carbon compounds, with varying nitrogen sources, by varying the percentage of the salts (especially N and P), by cultivation in oxygen rich air, cultivation at varying temperatures, in the light, in solutions, agar or on plaster of Paris, soaked in culture solution, etc. After two months cultivation they were examined as follows: the culture solution was tested with $FeCl_3$ upon phenolic compounds, so was the alcoholic or acetonic extract of the fungus mycelium, moreover this extract was evaporated on a watch-glass to detect crystalline substances and to the same purpose a small piece of mycelium was sublimated in a KLEIN–WERNER micro-vacuum sublimation apparatus.

In none of these ways we could detect in any culture the presence of a lichenic acid or similar substances. This was the most astonishing for the fungus C, which had been isolated from the heavy lichenized, soredial algal cover. By extracting this cover with acetone, evaporating the extract on a watch-glass and after washing away fats and chlorophyll with benzene I obtained a crystalline product, which could be purified by crystallizing from alcohol or ether. This substance shows the following characteristic colour-reactions: the so called homo-fluorescein reaction, characteristic for orcin derivatives, to wit the red colour with NaOH and chloroform and the green fluorescence after pouring this solution into water, furthermore a red colouration with $FeCl_3$, a yellow colour with KOH, a brownish-red colour with concentrated H_2SO_4, a yellowish-brown colour with paraphenylenediamin, a yellow colour with benzidin and with anilin. The substance becomes brown at 240° C. and carbonizes at 260° C. It crystallizes as short, white needles. Most probable it is identical with salazinic acid or an allied lichenic acid. Yet we were not able to detect this substance, which is easily recognizable by its marked colour-reactions, in any of the cultures of the fungus symbiont isolated from this cover.

Moreover it appeared that some proto-pleurococcoid algal covers too (namely those consisting of Apatococcus minor EDL.), though lichenized in a relatively small degree, were in the possession of a remarkable metabolic product, which may be isolated as follows:

the algal cover, scraped from an old stone wall is extracted in an extraction funnel with acetone; the acetone is destilled off and the remaining product vigorously shaken with benzene and filtered through a glass filter. In the filter remains the rough product, which may be purified easily by crystallization from alcohol or from ether. It is a beautiful white substance, consisting of very long, threadlike crystals, melting at 139° C., soluble in most organic solvents, best in chloroform and pyridin, less in hot alcohol, acetone or ether, slightly in these solvents when cold. No colour-reactions could be detected. It is without any doubt identical with the "acide phycique", discovered by LAMY in 1857, which substance was chemically investigated in the laboratory of Prof. Dr. G. VAN ITERSON by J. G. VAN DE SANDE (1927 unpublished). We have tried to establish the chemical constitution of this compound, making use of VAN DE SANDE's data for comparison.

The empirical formula of the substance followed from elementary analysis performed by Mr. P. J. HUBERS, laboratory for Organic Chemistry, University of Amsterdam, which showed the following values in two determinations:

C 69.22 %	H 11.33 %	N 3.64 %
C 69.78 %	H 11.37 %	N 3.68 %

From this we calculate the tentative formula as $C_{23}H_{45}O_4N$. This formula should yield C 69.13 % H 11.35 % N 3.51 %. Two molecular-weight determinations (melting-point depression in Camphor according to RAST) yielded 432 and 396 (calculated for $C_{23}H_{45}O_4N$: 399.59). VAN DE SANDE calculated from his determinations elementary formulae with yet more C and H atoms.

The substance which we shall provisionally name "apatococcin" shows, in alcoholic solution, a neutral reaction and cannot be shaken out of its chloroform-solution either by bases or by acids. Boiling with a dilute alcoholic solution of NaOH, or the action of cold concentrated NaOH, saponifies the substance. A foamy solution ensues, which, after acidification, yields a crystalline precipitate, consisting of very long, threadlike needles, m.p. 160° C. The elementary analysis of this product by Mr. P. J. HUBERS showed:

	C 68.28 %	H 11.26 %	N 3.94 %.
Calculated for $C_{21}H_{41}NO_4$:	C 67.89 %	H 11.13 %	N 3.77 %
$C_{22}H_{43}NO_4$:	C 68.53 %	H 11.24 %	N 3.63 %

Probably, therefore, in the original apatococcin an acid group was esterified, either with CH_3 or C_2H_5. A methoxyl and ethoxyl determination in this product by Mr. HUBERS yielded 9.50 % OC_2H_5 or 6.55 % OCH_3 (calculated for one group OC_2H_5 11.28 % and for one group OCH_3 7.77 %). Therefore, the presence of one group $COOCH_3$- (or $COOC_2H_5$) in apatococcin appears probable.

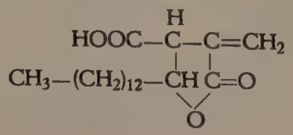
The substance seems saturated, in solutions neither permanganate nor bromine are decolorized (already stated by VAN DE SANDE). As to the position of the nitrogen it may be stated that the substance has neither alcaline nor alcaloid character as it does not form salts with dilute or strong acids and does not give any alcaloid-reactions in solutions. The nitrogen cannot be removed by saponification and action of HNO_2 does not seem to change the substance. It is, therefore, neither a simple acid-amid nor a primary or secundary amin (which is in good accordance with the data of VAN DE SANDE).

By action of phenylhydrazin the substance remains unchanged so that most probably it does contain neither an aldehyde nor a ketone group.

According to VAN DE SANDE the substance can be acetylated by boiling with acetic anhydride and a trace of sodium acetate for some days. In repeating this experiment, however, most of my substance deteriorated.

From the elementary formula it is probable that apatococcin possesses a long paraffin chain, which also would account for the foamy character of the sodium salt.

Though the investigation will be continued we expect that the substance may show a relationship to certain lichenic acids such as protolichesterinic acid (ASAHINA 1939), which acid, however, does not contain any nitrogen.



protolichesterinic acid.

In an apatococcus cover the substance is easily recognizable by the very characteristic curved threadlike crystals obtained on sublimation (fig. 1).

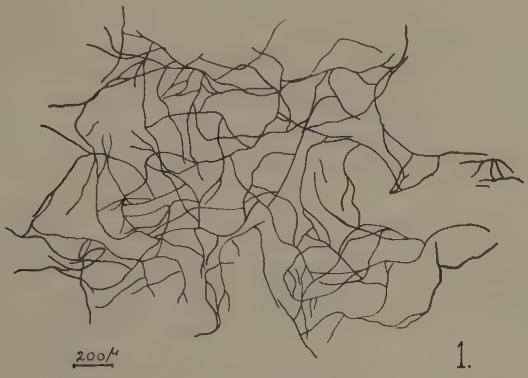


Fig. 1. Camera-lucida drawing of a sublimate from an apatococcus cover.

As, moreover, this substance could not be detected in any of the fungi, a number of pure cultures of the alga Apatococcus minor Edl. were sublimated in a KLEIN–WERNER apparatus. Initially the results were negative, but finally the crystals were clearly visible in the sublimate from some old cultures, which had partially died off from the extreme heat during the summer months (fig. 2). In a micromelting-point apparatus the melting-point of some of these crystals could be determined as = 129° C., which agrees reasonably well with the melting-point of "apatococcin" (139° C.), when we take into account the impurity of a sublimate like this. The double refraction of the crystals is too weak to be useful as a characteristic.

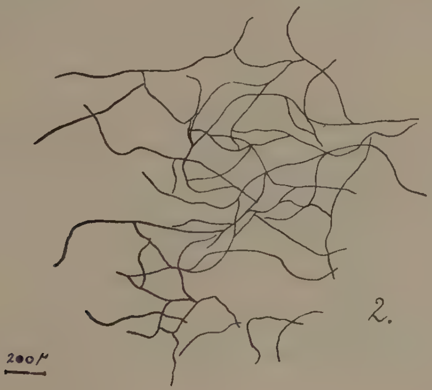


Fig. 2. Camera-lucida drawing of a sublimate from a dead pure culture of Apatococcus minor Edl. (cultivated on BEIJERINCK agar with 2% glucose). Both sublimates were carefully washed with water.

So it is very probable that the "apatococcin" is a metabolic product, which is made by the alga Apatococcus minor Edl. without the help of its fungal symbionts, a result which was to be expected, when we take into account the domination of the alga Apatococcus over the fungus in the covers and the specificity of apatococcin to Apatococcus covers, though these covers are in the possession of different, non specific fungi. Though it does not appear to be the alga which functions as a gonidium in most lichens, the results mentioned point strongly in the direction that, in considering the lichenic acid problem, we have to pay more attention to the algal part of the lichen than we were apt to do.

It was already a well-known fact that many lichenic acids consist of a lichenic acid s.s. (e.g. lecanoric acid) esterified with erythritol, which was already known as the metabolic product of the alga. Here we have an indication that some other substances as well may have an algal origin.

In some preliminary experiments we added apatococcin to cultures of some of our fungi on media poor in nutritive substances. No reaction was observed, the fungus being apparently unable to use this substance in its metabolism.

Conclusion.

The fungal symbionts in lichenized algal covers can be cultivated with more succes than true lichen fungi. Their great similarity to the latter makes it probable that they are related to certain true lichen fungi and that this alga-fungus symbiosis is comparable to the lichen symbiosis. In consequence they form an excellent object for the study of this symbiosis. The fungi are unable to fix atmospheric nitrogen. They cannot develop without aneurin, which they obtain, in nature, from their algal partner. In none of the cultures on various media, the presence of lichenic acids or similar products could be detected. On the contrary, it appeared that the alga Apatococcus is the producer of a remarkable metabolic product, called apatococcin, with the tentative formula $C_{23}H_{45}O_4N$. Some chemical properties of this substance are described. A relationship with certain lichenic acids is suggested. The investigation is continued.

I want to thank Prof. Dr. G. VAN ITERSON for his valuable help and for allowing me to make use of the unpublished work of VAN DE SANDE and Prof. Dr. F. KÖGL who kindly gave some critical remarks as to my work on the chemical constitution of apatococcin.

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Botany. — *On the influence of Colchicin upon the anthers of Carthamus tinctorius L.* By Miss J. M. KRIJTHE (from the Laboratory of Genetics, Agricultural Institute, Wageningen). (Communicated by Prof. L. G. M. BAAS BECKING.)

(Communicated at the meeting of February 28, 1942.)

Although the literature on the influence of colchicin on living matter is voluminous, only a few papers deal with the effects of this substance upon flowers and inflorescences. Some of these papers only mention morphological characteristics such as pollen- or stomatal size, from which measurements often deductions are drawn as to tetra- or polyploidy of the material, often without cytological control.

Adequate cytological research has been published by LEVAN (1939), WALKER (1938), DERMEN (1938) and SATÔ (1939) — all on monocotyledons. The above authors followed — with minor variations — the following procedure; the entire inflorescence was treated for 5—6 days with a colchicin-solution of 0.1—1 %. Attention was almost exclusively directed towards changes in nuclear division, to wit: the absence of the spindle and chromosome-pairing (the chromosomes, however, dividing), with the subsequent absence of cell-division, by which absence abnormal large cells appear. These cells either show a large, tetraploid nucleus or several small nuclei.

This may be demonstrated not only with pollen grains, but also with unicellular staminal hairs. SATÔ mentions the appearance of irregular and incomplete cell-walls, without detailed description of their nature.

Material.

The present paper deals chiefly with phenomena observed in the inflorescences of the safflower (Carthamus tinctorius L.).

The safflower, a composite belonging to the Cynareae, appeared to be a favourable object because of its short vegetation-period (3—4 months), its profuse flowering (30 inflorescences per plant) and the relatively small number of chromosomes (haploid 12).

Method.

It was originally attempted to obtain tetraploid plants by the treatment of seeds and young seedlings with colchicin. As this proved to be unsuccessful (only two pairs of leaves developing subsequent to the treatment showing effects), young inflorescences (3—5 mm cross-section) were used.

The involucre was pushed aside by means of pincers, after which the cavity above the individual flowerets was filled with a colchicin-agar (0.4—0.8 % colchicin), or an aqueous solution of colchicin (10 drops aqueous 0.2 % solution) was applied for three consecutive days. Controls received 1 % agar, or water.

The involucre closed after treatment. The controls showed normal growth. The effects described seem, therefore, due to the colchicin applied.

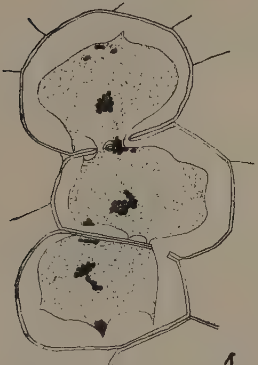
The treated inflorescences were enclosed for three days in parchment bags, to prevent dessication. The fixation of the flowerets took place either in NAVASHIN's or CARNOY's fluid, between 7.30 and 11.30 a.m. The sections were cut to a thickness of 10 μ and stained with HEIDENHAIN-haematoxylin or with gentian-violet.

Results.

1. *Morphological changes.*

Already after one week a broadening of the entire inflorescence could be observed.

The normal inflorescence is pyriform, the treated inflorescence showed a flat apex and a sudden transition towards the petiole. The tips of the involucral leaves curled inwards, instead of remaining in their vertical position.



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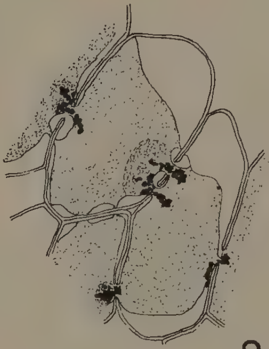

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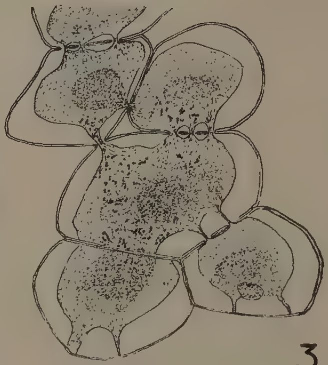

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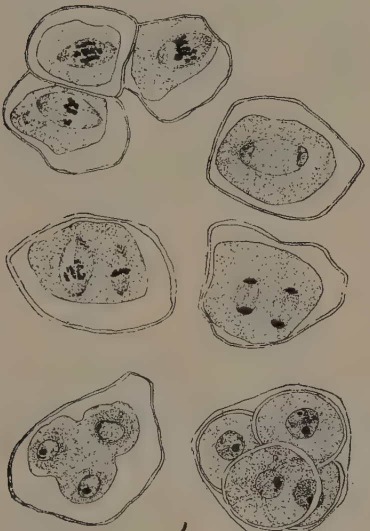

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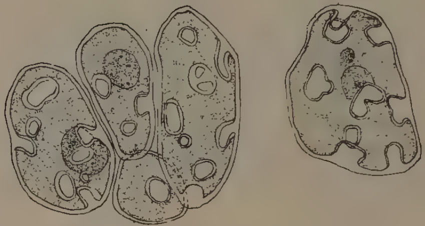

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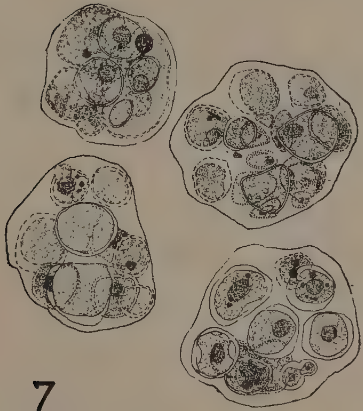

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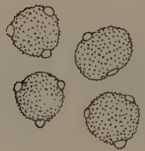

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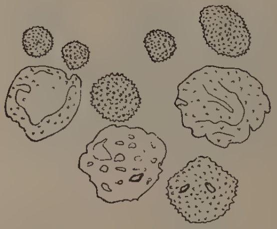

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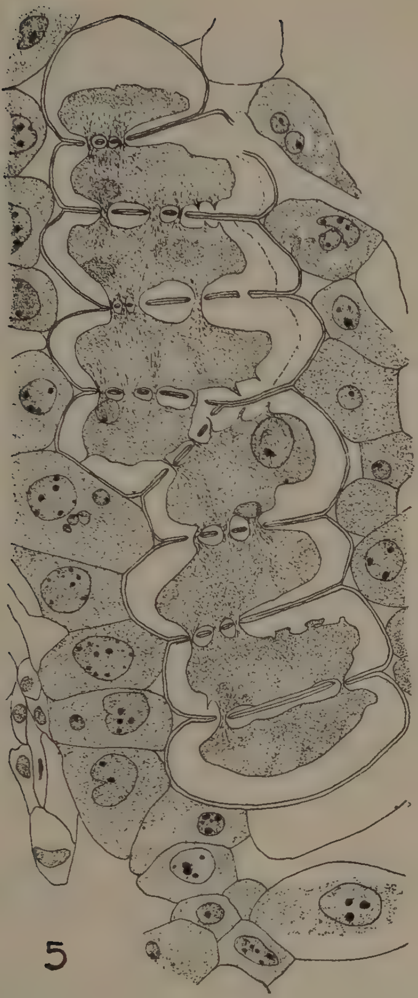

LEGEND TO FIGURES.

Fig. 1. Archespore in division, chromosomes within protoplasm, partly in the plasmodesms. 500 ×.

„ 2. As fig. 1 chromosomes all in plasmodesms. 500 ×.

„ 3. Chromosome-fragments passing through openings in the wall. 500 ×.

„ 4. Normal pollen-formation. Four nuclei are present prior to wall-formation. Spindles apparent. 500 ×.

„ 5. Theca, longitudinal, showing plasmatic connections between pollen mother cells. Note lobed nuclei in tapetum. 500 ×.

„ 6. Pollen mother cells showing wall-intrusion. Protoplasm retracted from wall. 500 ×.

„ 7. Pollen mother cells showing various number of pollen grains. 500 ×.

„ 8. Normal pollen grains. 135 ×.

„ 9. Treated pollen grains. 135 ×.

Further development showed a progressive unfolding of the involucre, showing the flowerets at the base of the inflorescence. The untreated inflorescence remained covered. Treated flowers were 4—5 weeks late in bloom. Untreated flowers show the protrusion of the corollar tube from their involucre, while the pointed petals are in a horizontal position. The floweret reaches a length of 20—25 mm. The long slender gynaecium shows two yellow stigmata, densely covered by stiff hairs, pointing downwards. The pollen-grains are round, with four pores, bright yellow in colour and homogeneous in size. The yellow corolla becomes orange after flowering.

Treated flowers reach a length of ± 10 mm, do not protrude outside the involucre. The corollar tube is strongly wrinkled, while the petals are ribbon-shaped, with a blunt apex. The petals do not open during flowering. The contents of anthers seem partly desiccated and the short, heavy gynaecium carries a single, heavy, stigma, showing irrigular hairs, pointing in all directions. With the unaided eye the stigma shows a woolly effect. The pollen is irrigular in size, dark in colour and with a variable number of pores. No seed-formation takes place. Treated buds are shorter and are shaped as an inverted cone, while the controls show buds which are long and slender and of a conical shape. During the period of flowering treated flowerets are dark orange, much darker than controls after their period of flowering.

2. *Cytology.*

Dependent upon the stage of development the colchicin shows different effects. The division of the archespore is influenced in another way as the reduction-division. In all divisions the absence of a spindle, as observed by other authors, was apparent.

The chromatin seems scattered throughout the protoplasm in an arbitrary way. Sometimes the chromosomes are packed into dense clusters, some sections showed a number of small darkly stained granules; possibly chromosome-fragments. As far as could be ascertained, the number of chromosomes remained normal, while the resting nuclei showed no aberrant size. Controls show the normal scheme of division, the spindle being clearly visible. The occurrence of large, lobed nuclei in the tapetum was also observed in the controls and seems, therefore, to be a normal phenomenon.

The above phenomena are in harmony with the findings of other authors. The literature, however, seems silent on the following point; the influence of colchicin upon the cell wall.

The pollen mother cells show openings in their walls at the points of contact with neighbouring cells. Where the section was made centrally through such an opening, smaller or wider protoplasmic strands could be observed, connecting the plasma of the adjacent cells. Very young, not yet thickened walls already show these pores or pits, which are much wider than those known of the plasmodesms. In some cases all of the pollen mother cells over the entire length of the anther are joined by strands of protoplasm. Untreated anthers never showed these connections. Moreover, the shape of the wall in the neighbourhood of the pit and, the topography of the protoplasm in this region, showed that the structures are no artefacts.

Colchicin showed another influence upon the pollen mother cells already formed. Here the cells are often rounded, while the wall often grows to such dimension that hardly any lumen is left. The dimensions of the cells show them to be pollen-mother cells and not pollengrains. Apart from this phenomenon the cell wall may become thickened at localized spots, which seem scattered over the surface of the pollen mother cell. The substance which is deposited in the above cases seems to be callose. Reso-blue gave a beautiful colour, while no birefringence in polarized light could be observed.

The formation of these "wall-intrusions" seems to bear no connection to cell division. Their arbitrary distribution seems to corroborate this. In some cases in a single theca the mother cells formed apparently normal tetrads at one end, while at the other end the cells showed wall-intrusion.

Reduction division under the influence of colchicin shows that, instead of four pollen grains, 10—17 pollen grains appear from a single pollen mother cell. Most of the grains showed the presence of a nucleus, the smallest grains excepted. In these too little chromatin was probably present.

The "warty" appearance of the normal pollen is less evident in the "treated" grains. Pores are indicated in the larger ones, which are often furrowed. Pores seem to be entirely absent from the grains of normal and subnormal size.

Discussion.

Pollen-formation in Monocotyledons involves the formation of a wall after the heterotypic division (dyad) as well as after the homoiotypic division (tetrad). In the Dicotyledons pollen formation is simultaneous. After the termination of the entire reduction division, the four nuclei are situated at the apices of a tetraedron, the wall is formed by infolding of the wall of the pollen mother cell. In the Monocotyledons the reaction is nuclear, in the Dicotyledons it is plasmatic.

The difference in reaction between Carthamus and Allium (LEVAN) might be ascribed to the above facts. The formation of pollen mother cells in Carthamus with irregular wall-intrusions might be a link in the process, terminating in the formation of supernumerary pollen grains.

In regard to the formation of the cell-wall intrusion many instances are known where such abnormal phenomena occur. Orchid mycorrhiza causes abnormal thickening of the walls of the host-cells (BURGEFF, 1932).

In old algal cultures wall-thickening has been observed (KÜSTER, 1935). MICHAELIS (1926) obtained supernumerary pollen grains from pollen mother cells by cold-shock (of 20° C). Complete walls were always formed, however, in this case.

Structures, analogous to those observed in Carthamus anthers are, of course, sieve tubes and storage cells in endosperms. In these cases, however, the plasmodesms are much narrower. Vascular wound tissue (TIMMEL, 1927) and centrifuged Spirogya filaments (VAN WISSELINGH, 1903, 1909) show incomplete walls.

A striking resemblance of the structures observed with the intercellular plasmatic connections of Rhodophyceae (JUNGERS, 1933) cannot be denied.

Only a few references were found in the literature pertaining to the influence of chemicals upon wall-formation. NĕMEC (1904) observed, in the roots of Vicia faba, incomplete wall-formation with concomitant absence of the spindle after treatment with chloral-hydrate. NAVASHIN (1938) treated seeds of various plants with sublimated acenapthene. He observed the formation of new cell-walls within the old wall, dividing the cell into smaller cells, some of them anucleate. As acenapthene seems to assert a similar (although weaker) influence as colchicin upon plant-cells, this result seems interesting.

SUMMARY.

Young inflorescences of Carthamus tinctorius L. were treated for three consecutive days with colchicin-agar (0.4—0.8 % colchicin) or a solution of colchicin 0.2 %. Dependent upon the stage of development of the cells the reaction was different. In all stages of the development of the archespore the spindle remained absent. Pollen mother cells during their formation communicated by means of protoplasmic strands running through large openings in the walls. Mature pollen mother cells showed, at arbitrary places of their walls, callose intrusions. By these intrusions the mothercells are divided into 10—17 pollen grains.

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Anatomy. — *Biologic-anatomical Investigations on the Bipedal Gait and Upright Posture in Mammals, with Special Reference to a Little Goat, born without Forelegs.* I. By E. J. SLIJPER (Utrecht). (From the Institute of Veterinary Anatomy of the State University, Utrecht, Holland; Director Prof. Dr. G. KREDIET).

(Communicated at the meeting of February 28, 1942.)

I. *Introduction.*

In June 1939 our institute received a little he-goat, three months old, born without forelegs. On the left side it had only a scapula, just as GRAU (18) described of a new-born horse. This bone terminated in a little knob, which can be explained as the synostosis of the scapula with a vestigial humerus [see MURRAY (43) and GRAU (18) in opposition to JENNY (27)]. On the right side the animal possessed a very small and highly deformed little leg with a hoof, in the same way as has been described by GRAU (18) of a goat. Malformations of this kind are not uncommon at all; in our institute there are skeletons of new-born and young calves, goats and dogs without forelegs. They have for example been described by FULD (17; dog), REGNAULT (50; dog), GRAU (18; horse, goat), JENNY (27; sheep) and LESBRE (33) of man and all the domestic animals. Most of these animals were very well capable of living; REGNAULT (50) possessed a bipedal dog that was twelve years old. Unfortunately our little goat died at the age of one year owing to an accident. The first seven months of its life it passed its days on the grass-field, moving forward by jumps on its hindlegs in a semi-upright posture. The body made an angle of nearly 45° with the ground and the hoofs of the hindlegs were placed much farther forward under the body than in a normal goat, in order to bring the supporting surface under the centre of gravity. The manner of locomotion was quite similar to that of a jumping-hare or a kangaroo, both hindlegs leaving the ground at the same time. During the winter the animal lived in the stable.

Almost every author who has described animals born without forelegs, or animals whose forelegs had been amputated [see for example FULD (17; dog), COLTON (10; rat), KOWESCHNIKOWA und KOTIKOWA (32; cat), JACKSON (26; dog)] has restricted himself to an investigation of the vestigial foreleg or a few characteristics of the hindleg. In order to make a more intensive use of the material, the researches on my bipedal goat were connected with a biologic-anatomical investigation on the changes that have taken place in the locomotor-apparatus of mammals with a bipedal gait or an upright posture. So on the one side it was possible to give a better explanation of some characteristics of the bipedal goat, on the other side the changes in the structure of this animal served as a kind of proof of the explanation of the phenomena common to several or all bipedal and upright mammals.

On the whole these animals belong to three orders of mammals: the *Marsupialia*, *Rodentia* and *Primates*. If possible in each of this three orders I examined a skeleton of a representative of one of the following five types of locomotion (fig. 1): 1st. A walking or running animal [*Thylacinus cynocephalus* (Harris), *Lepus europaeus* Pall.]. 2d. An animal that walks on the branches of trees or that climbs with all four extremities [*Trichosurus vulpecula* (Kerr), *Phascolarctos cinereus* (Goldf.), *Sciurus vulgaris* L., *Cebus apella* L., *Trachypithecus pyrrhus* (Horsf.); for the kind of locomotion see MUYBRIDGE (44), BÖKER (5), SLIJPER (61)]. When climbing these animals now and then show an upright posture. 3d. A bipedal jumping animal [*Dendrolagus inustus* Müller u. Schlegel, *Bettongia lesueuri grayi* (Gould), *Macropus giganteus* (Zimm.), *Pedetes caffer* (Pall.), *Jaculus jaculus* (L.); for the kind of locomotion see BÖKER (5), SLIJPER (61), MUYBRIDGE (44), ELFTMAN (14), HOWELL (25), HATT (21), LULL (36)]. These animals mainly jump by simultaneous propulsive strokes of both hindlegs. Their body makes an angle of nearly 45° with the horizontal plane, but is kept in balance on the hindlegs by the very long and heavy tail. At rest the tail serves as an adventitious support for the body-weight. 4th. A hanging-climbing animal. This type is only known of the Primates [I examined *Ateleus paniscus* (L.), *Hylobates lar leuciscus* Geoffr., *Pongo pygmaeus* (Hoppius); for the kind of locomotion see BÖKER (5), SLIJPER (61), PRIEMEL (49)]. These animals chiefly climb with their extremely elongated forelegs. When climbing their posture is nearly upright, they have no tail, but the body is kept in balance by the support of the forelegs. 5th. Man, the only real bipedal walker among the mammals. His posture is perfectly upright, there is no tail, but the body is kept in

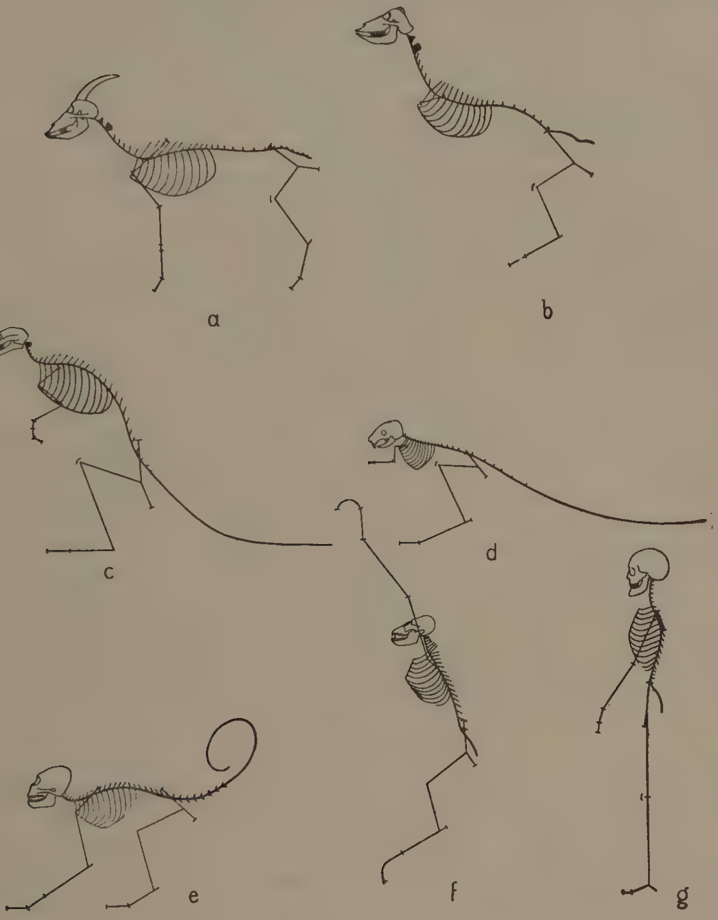

Fig. 1.

Schematic figures of the skeletons of the different types of mammals, described in this paper. *a.* Normal goat. *b.* Bipedal goat. *c.* Kangaroo (*Macropus giganteus* (Zimm.); bipedal jumping mammal). *d.* Jumping-mouse (*Jaculus orientalis* Erxl.; bipedal jumping mammal). *e.* Monkey (*Cebus apella* (L.); climbing mammal). *f.* Orang utan (*Pongo pygmaeus* (Hoppius); hanging-climbing mammal). *g.* Man (*Homo sapiens* L.; bipedal walking mammal).

balance on the hindlegs by the remarkable S-shaped vertebral column with a little aid of some muscles. The body-weight is supported entirely by the hindlegs; they move the body forward by alternating propulsive strokes. 6th. In connection with the upright posture of their body also some bats were included in the investigation.

Compared with the above-mentioned mammals the posture of the bipedal goat was a very unfavourable one. The body had to maintain a semi-upright posture without the aid of the counterweight of a tail or the support of forelegs. Moreover the total weight of the body was carried by the hindlegs alone, which were not stretched as in man but showed the same angles between the different parts as in most quadrupedal mammals. It was a priori to be expected, that these extremely unfavourable circumstances would cause some changes in the skeleton and the musculature. For the purpose of comparison I used a little female goat of nearly the same age. This control-animal was in a much better physical condition than the bipedal one. Its horns were already developed and it had five grinding teeth in stead of four. The chief measurements of the skulls were at a ratio of 84 to 100. All measurements of the control-animal therefore were converted into a ratio of 100 : 84. In table 1 is shown the difference of the dimensions of the skeleton between the two goats, calculated in % of the converted dimensions of the control-animal. Differences less than 6 % were always neglected.

In this paper special attention will be paid to the skeleton and musculature of the hindleg, the pelvis and the thorax. The changes that took place in the vertebral column and its musculature will be described in a special paper, devoted to the comparative anatomy of the body-axis of mammals.

At the end of this introduction I would like to express my most heartfelt thanks to Prof. Dr. CHR. P. RAVEN (Utrecht), Prof. Dr. H. BOSCHMA (Leiden) and Dr. G. C. A. JUNGE (Leiden), for the kind and obliging way in which they placed the material of their collections at my disposal. Grateful acknowledgement is also made to Dr. L. D. BRONGERSMA (Leiden) for the revision of the nomenclature and to Mr. W. WIJGA (Utrecht) for the correction of the manuscript.

II. *Hindleg.*

Neither the length of the whole leg, nor the proportional length of its separate bones, showed a marked difference from that of the control-animal (table 1, fig. 2). Only the first phalanx was a little bit elongated. This may cause no surprise, since in running

TABLE 1

DIFFERENCE OF THE DIMENSIONS OF THE SKELETON BETWEEN THE BIPEDAL AND A NORMAL GOAT

+6 (-6) means, that the dimension in the bipedal goat is 6% greater (lesser) than that of the control-animal, calculated in % of the converted dimensions (100:84) of this control-animal.

HINDLEG

	Length	Thickness of shaft	Dimensions of joint-surfaces			
			Proximal		Distal	
Femur	+ 5	+ 9	+32	+26	+21	+12
Tibia	+11	+20	+26	+21	+20	+31
Tuber calcanei	+19	+16				
Astragalus	+13	+19	+28	+18	+29	0
Metatarsus	+ 7	+ 6	+ 9	+18	+16	+28
1st Phalanx	+26	0	+18	+21	+18	+18
2d Phalanx	+ 6	+ 5	+17	+ 8		
Patella	+11	+ 7	+22	+11		
Collum femoris	+20					

PELVIS

Length of:	
Pelvis	+12
Ilium	+ 9
Ischium	+23
Pubis	- 5
Symphysis	+27
For.obturatum	+ 1
Distance betw. right and left:	
Tuber coxae	+ 5
Tuber ischii	+16
Acetabulum	-38
Breadth of:	
Ala ilii	+ 3
Sacrum at ilio-sacral joint	+13

Thickness of:		
Ilium	+41	+22
Acetabulum	+36	+36
Ischium	+21	+54
Pubis	+16	+82
Dimensions of joint-surface of acetabulum	+29	+24

THORAX

The bipedal goat possessed 12 ribs,viz.: 7 true and 5 false ribs. The control-animal had 13 ribs,viz.: 7 true and 6 false. Both animals had 12 bicipital ribs.

Length of sternum with (without) proc. xiphoideus: -4 (-9)
Breadth in the middle of sternebrae: 1st +103; 2d +38; 3d +29; 4th +32; 5th +38

mammals, to which the goat belongs, the proportional dimensions of the hindlegs and their different parts are nearly the same as in bipedal jumping mammals (table 2).

TABLE 2
PROPORTIONAL DIMENSIONS OF THE HINDLEG IN MAMMALS

Species	Length of bone in % of total length of hindleg					
	Femur	Tibia	Tarsus	Metatarsus	Phalanges	Tuber calcanei
Thylacinus cynocephalus (Harris)	36	38	6	11	19	
Epimys norvegicus Erxl.	31	37	4	13	15	6
Lepus europaeus Pall.	30	37	5	14	4	7
Canis familiaris L.	33	33	8	13	13	7
Average of walking mammals	32	36	6	12	13	7
Equus caballus L. (dom.)	27	27	7	23	16	8
Giraffa camelopardalis (L.)	19	21	4	40	16	5
CAPRA HIRCUS L. CONTROL	30	33	4	19	14	5
CAPRA HIRCUS L. BIPEDAL	30	34	4	19	13	6
Average of running mammals	25	27	5	27	15	6
Phascolarctos cinereus (Goldf.)	40	30	7	7	16	
Ursus arctos L.	36	29	4	10	11	5
Hippopotamus amphibius L.	42	26	10	10	14	13
Rhinoceros sondaicus Desm.	41	27	10	11	11	10
Elephas maximus L.	53	29	6	6	6	6
Average of heavy mammals	42	28	7	9	12	8
Didelphis marsupialis L.	35	37	8	8	12	
Trichosurus vulpecula (Kerr.)	34	34	7	7	18	
Sciurus vulgaris L.	30	38	7	7	18	4
Anomalurus beecrofti Fraser	35	35	5	9	16	3
Trachypithecus pyrrhus (Horsf.)	37	35	5	10	13	3
Average of climbing mammals	34	36	6	8	15	3
Dendrolagus inustus Müll.u.Schleg.	33	36	6	11	14	6
Bettongia lesueuri grayi Gould.	26	36	4	16	18	7
Macropus giganteus Zimm.	26	41	4	15	14	
Pedetes caffer (Pall.)	25	36	6	15	18	5
Jaculus jaculus (L.)	23	35	3	25	14	4
Av. of bipedal jumping mammals	27	37	5	16	16	5
Ateleus paniscus (L.)	35	33	7	11	14	5
Hylobates lar leuciscus Geoffr.	40	33	5	11	12	4
Pongo pygmaeus (Hoppius)	33	30	8	13	16	6
Av. of hanging-climbing mammals	36	32	7	11	14	5
Homo sapiens L.	44	38	4	8	6	5

The factors, determining the proportional length of the femur ad tibia have been discussed by several authors. REGNAULT (50) and FULD (17) found, that in dogs whose forelegs were amputated the femur proportionally was shortened and the tibia elongated. The differences between these dogs and the control-dogs, however, remained within the bounds of variability of the species, so that not too much value may be attached to their conclusions. On the contrary COLTON (10) found, that in bipedal rats the femur was elongated and that it showed the same length-ratio as in man. His explanation, however, that this phenomenon might be caused by the fact that both man and rat are plantigrade, the dogs on the contrary digitigrade walkers, is not in accordance with the fact that the amputated dogs of FULD just became plantigrade.

As is shown by the data collected in table 2, in bipedal jumping mammals on the whole the femur has been shortened, the tibia a little elongated and the metatarsus very much elongated [see also HOWELL (25), SCHUMANN (59), LYON (37) and MÜLLER (41)]. In heavy quadrupedal mammals (especially in heavy Ungulates) as well as in hanging-climbing Primates and man, the femur is elongated and the tibia and metatarsus are shortened [see also BÖKER (5), GREGORY (19) and others]. In both types of mammals the weight supported by the hindlegs is increased. In adaptation to this increase of weight in heavy quadrupedal mammals, in man and (to a lesser degree, however) in anthropoid apes, the femur has a more or less vertical position, while in bipedal jumping mammals its position is oblique or even nearly horizontal [*Macropus*; see ELFTMAN (14)].

If the weight supported by the hindlegs is increased, then the femur is shortened if it has an oblique position, while it is elongated if its position is more or less vertical. This is easy to understand, because in a vertical position of the femur the distance between tuber ischii and knee-joint (length of the hamstring-muscles) is much shorter than in a horizontal position. The shortening of the fibers of the hamstring-muscles therefore must be compensated by the increase in length of the femur. On the other hand in mammals with a more or less horizontal femur this cannot be elongated, since in that case the transmission of power would be too unfavourable [see also AICHEL (1)]. So the differences between the bipedal dogs and rats could perhaps be explained by the fact, that in dogs the femur is as long as the tibia, while in rats it is shorter. It is also possible, that the position of the femur in the dogs differed from that in the rats. Unfortunately the publications of FULD and COLTON do not give exact information on this point.

The elongation of the first phalanx in the bipedal goat (table 1) may be connected with the fact that the feet had to be placed much more forward, in order to bring the supporting surface under the centre of gravity. As is shown in fig. 2 the more horizontal position of the metatarsus and the toes, required by the above-mentioned forward motion of the supporting surface, caused a change in the direction of the calcaneus and especially of the tuber calcanei. For in digitigrade and unguligrade mammals the direction of the tuber calcanei is always nearly parallel to that of the femur. This position guarantees the most favourable effect for the contraction of the m. gastrocnemius and flexor digitalis sublimis. In the normal goat there is an angle between the tuber calcanei and the metatarsus (fig. 2a). In the bipedal goat the position of the tuber calcanei with regard to that of the femur was nearly the same as in the control-animal. In connection with the altered position of the metatarsus, however, the tuber calcanei showed the same direction as this bone (fig. 2b). In consequence the angle between tuber calcanei and tibia was so much smaller than in the control-animal that the tarsal joint could not be completely stretched. The elongation of the tuber calcanei lengthened the lever of the tarsal joint.

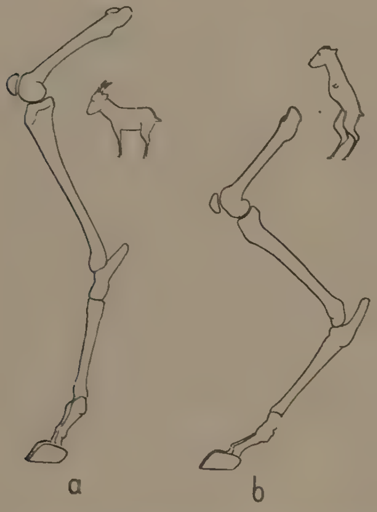

Fig. 2.
Left hindleg of the normal (*a*) and the bipedal (*b*) goat.

Neither an enlargement of the trochanter maior [RUDOLF (52)] or a limitation of the movements in the joints to one single plane, nor a reduction of the fibula [SCHAPIRO (55), SCHUMANN (59), LYON (37), HOWELL (25)] or of the number of toes could be expected in our goat, since these characters are already common to running and bipedal jumping mammals [SLIJPER (61)].

The very marked increase in thickness of all the bones of the hindleg and especially the increase of their proximal and distal ends and their joint-surfaces, without doubt have been caused by the increase of the weight supported by this leg. This is in perfect accordance with the considerations of WEIDENREICH (66) and the experiments of WERMEL (68) and STIEVE (62), but it does not quite agree with the data given by FULD (17) and COLTON (10). In the dogs of FULD the bones of the hindleg were not thickened, but only the quantity of bone had been augmented. The joint-surface of the distal epiphysis of the tibia of the dog of FULD was diminished, but in the rats of COLTON it had been increased. In the dogs the acetabular joint-surface too was diminished, while in the goat it has been increased, just as RUDOLF (52) has shown for the kangaroo. The elongation of the collum femoris is an adaptation to the narrowing of the pelvis at the acetabulum (see chapter III, sub 7).

The preservation of the muscles did not permit me to compare their weight with that of the control-animal. In general, however, it can be said that the greater part of the muscles in the bipedal goat were better developed than in the quadrupedal one, with the exception of the psoas-musculature, which showed a minor development. On the whole this observation agrees with the data given by FULD (17) for a bipedal dog, KOWESCHNIKOWA und KOTIKOWA (32) for a bipedal cat and by ALEZAIS (2), SCHAPIRO (55), ELFTMAN (14) and HOWELL (25) for bipedal jumping mammals. The psoas-musculature will be dealt with in chapter III.

III. *Pelvis.*

1. **General remarks.** On the whole we may consider the structure of the pelvis as a compromise between the demands made by statical and mechanical forces, by the organs of the pelvic cavity, which have to take up a certain space and by the insertions of the muscles. An investigation into the differences in the structure of the pelvis thus must take into account these four demands and may not be restricted to one or two of them as ELFTMAN (14), BYKOV and KOTIKOWA (9) and many investigators of human anatomy have done [for a detailed discussion of the literature see ARIENS KAPPERS (3)].

As the pelvis is not a simple perpendicular pillar of the body-axis, the statical force (that is the gravitation) may be resolved into several different components [see for example MIJSBERG (45)]. The most important of these components are: 1st. A force going from the ilio-sacral joint through the ilium to the acetabular joint, where it is compensated by the counter-pressure of the supporting leg. Direction, length and thickness of the ilium, as well as the thickness of the acetabulum and the surface of the acetabular joint may be influenced by this force. 2d. A force that tries to rotate the right and left halves of the pelvis in an upward and outward direction. In future this force will be called the exorotation. In the first place, this exorotation is caused by the fact, that the point where the femoral head supports the acetabulum lies laterally to the ilio-sacral joint. In the second place it is caused by the rotation of the vertebral column round the transverse axis of the ilio-sacral joint. This rotation is caused by the weight of the body: the lumbar vertebral column tries to move downward, the sacrum tries to move upward. This bone transfers the upward force to the ischium by the broad ligaments and the lig. sacro-(caudo-) tuberosum. So it causes the exorotation of the ischium. The exorotation is compensated by the pubis and the symphysis pelvis. If the symphysis has been sawn through, the halves of the pelvis turn aside [FENEIS (15)]. The size of the exorotation-force depends on the position of the acetabulum, the divergence of the ischia and the size of the body-weight that rests upon the hindlegs.

In man these factors would be augmented by an outward directed component of the body-weight in the ilio-sacral joint. The existence of this component would depend on the fact, that at the ilio-sacral joint the caudal border of the ala sacralis is narrower than the cranial one. In consequence of this fact the sacrum would act as a kind of coping-stone in an arched roof [MEYER (39), BRAUS (6)]. Recently LÜHKEN (35), however, has shown, that this component does not cause an outward directed force at the symphysis pelvis but on the contrary an inward directed one. But his conclusion, that the symphysis in man has to resist pressure in stead of tension cannot be right, as FENEIS (15) has shown, that tension is prevalent in the symphysis of man. Moreover LÜHKEN has neglected the other exorotating forces. It is possible, however, that the symphysis of man has to resist less tension-force, than that of other mammals especially of other Primates. The fact, that man has a fibro-cartilagineous symphysis instead of a bony one, as well as the comparatively low symphyseal index (see table 3), might be an indication of this opinion. The theory of the coping-stone does not hold with regard to other mammals, because their sacrum does not rest upon the pelvis but is hung from the ala ilii [BRUHNKE (8)].

The mechanical forces, caused by the locomotion of the animal are the same as the above-described statical ones. Additional forces are the reciprocal shifting of the two halves of the pelvis in mammals that walk by alternating strokes of their hindlegs, as well as the rotation of the pelvis in a dorsal direction caused by the shock when the foot is planted on the ground. The first force is compensated by the symphysis, the second by the ligaments of the ilio-sacral joint, the tension of the m. rectus abdominis [STRASSER (63)] and the tension of the m. psoas minor. The capacity of the pelvis is influenced by the bulk of the different organs, the size of the faeces, but especially by the dimensions of the foetus at birth [ELFTMAN (14)]. This is illustrated by the different dimensions of male and female pelves [SCHMALTZ (56), BRAUS (6)] and by the fact that some sexual hormones are able to alter the structure of the pelvis [RUTH (54), HISAW (24), HAWRE, MEYER and MARTIN (23)]. Under the insertions of muscles that influence the structure of the pelvis, special attention must be paid to the m. glutaeus medius (length and width of ala ilii), the hamstring-muscles (length of ischium) and the adductor muscles (length of symphysis).

2. **Thickness of bones.** As already has been shown for the hindleg, it is not surprising at all that the increase of the weight supported by the pelvis has caused a considerable thickening of all bones, but especially of the acetabulum and pubis. The acetabular joint-surface too is enlarged to a very marked degree (fig. 3, table 1).

3. **Position of the pelvis.** Since the sacral vertebra principally transmits the power from the lumbar vertebral column to the pelvis and reciprocally, the angulus ilio-lumbalis has proved to be a safer indication of the different forces acting on the pelvis than the angulus ilio-sacralis, which has been determined by MIJSBERG (45), NAUCK (46) and others. The angulus ilio-lumbalis is the angle between the ilium and the axis of the lumbar vertebral column that is produced in a caudal direction. The angulus sacro-lumbalis is the angle between this caudally produced axis and the axis of the sacrum. In quadrupedal mammals a more or less vertical position of the ilium is the most favourable to support the body-weight, while a more or less horizontal position is the most favourable for the transmission of the locomotor-power from the hindleg to the vertebral column. In consequence the heavy quadrupedal mammals show a comparatively wide ilio-lumbar angle, while it is comparatively narrow in smaller or lighter mammals, especially in those species that have a more or less jumping locomotion (*Leporidae, Felidae*; see table 3).

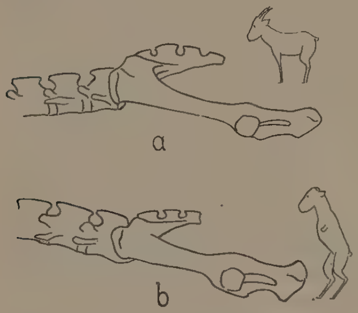

Fig. 3.
Lateral view on the pelvis of the normal (*a*) and the bipedal (*b*) goat.

The direction of the vertebral column in upright going and bipedal mammals makes it possible to combine a nearly vertical ilium with a very narrow ilio-lumbar angle [see table 3, especially the Primates; see also NAUCK (46), PRIEMEL (49)]. In the normal goat there is an ilio-lumbar angle of 23°. In the bipedal one this angle had been reduced to 0° (fig. 3). The angle between the ilium and the horizontal plane amounted from 25° to about 65 à 75°.

4. **Position of the sacrum.** In literature one can find several different explanations for the position of the sacrum and especially for its position in man, which is characterized by the possession of the remarkable promunturium. Several authors try to explain the nearly right angle between the lumbar and sacral vertebrae in man by the tension of the dorsal back-musculature or its demands for insertion [LE DAMANY (12)]. Other investigators on the contrary believe, that it is the demand of space in the pelvic cavity

that determines the position of the sacrum. Recently BLUME (4) has shown, that the development of the promunturium is in no way connected with statical or mechanical forces. From the data given in table 3 it is evident now, that the width of the angulus lumbo-sacralis (see sub 3) completely depends on the other factors determining the capacity of the pelvic cavity. So it can be seen that a wide lumbo-sacral angle occurs in those mammals that have a long sacrum and a narrow ilio-lumbar angle [compare for example *Ursus arctos* L. (6 sacral vertebrae) with *Panthera leo* (L.) (2 sacral vertebrae, same ilio-lumbar angle) or *Bos taurus* L. (ilio-lumbar angle of 30°) with *Rhinoceros sondaicus* Desm. (70°)]. The possession of a promunturium is not limited to man, but it occurs in several different quadrupedal mammals (*Sus, Dicotyles, Haplomys, Ursus*). Among the bipedal mammals a widening of the lumbo-sacral angle only occurs in the *Macropodidae*. For in the other species there was no marked change of the ilio-lumbar angle or the number of sacral vertebrae. The increase of the number of these vertebrae in Primates certainly has been the chief factor that caused the widening of the lumbo-sacral angle.

The above-mentioned opinion is fully borne out by the fact, that in the bipedal goat the ilio-lumbar angle decreased by 23° while the lumbo-sacral angle increased by 16° (see also table 1 and fig. 3).

TABLE 3. POSITION AND PROPORTIONAL DIMENSIONS OF THE PELVIS IN MAMMALS.

Species	Dimensions in % of the distance between occipital crest and cranial border of sacrum									Situation of pelvic inlet with regard to the sacral vertebrae 1)	Number of sacral vert.	Angle between (in degrees)		
	Length of bone					Breadth at			Height of pelvis					
	Ilium													
	Presacral	Postsacral	Total	Ischium	Symphysis	Pelvic inlet	Acetabulum	Tuber ischii				Ilium and lumbar vertebrae 2)	Sacrum and lumbar vertebrae	Ilium and pubis
Thylacinus cynocephalus (Harris)	6	6	12	10	12	7	6	7	6	c.2S.	2	28	0	117
Lepus europaeus Pall.	8	8	16	12	8	6	6	9	7	c.2S.	2	8	20	105
Panthera leo (L.)	7	8	15	11	12	8	6	8	6	c.2S.	2	15	15	130
Canis familiaris L.	7	7	14	9	7	7	6	11	9	c.3S.	3	18	35	135
Ursus arctos L.										5S.	6	12	56	140
Sus scrofa L. (dom.)	6	9	17	11	9	9	8	8	7	c.3S.	3	15	50	105
Average of walking mammals	7	8	15	11	10	7	6	9	7			15	29	122
CAPRA HIRCUS L. CONTROL	4	14	18	11	8	9	9	9	9	c.4S.	4	23	36	140
CAPRA HIRCUS L. BIPEDAL	4	13	17	14	10	10	4	9	7	c.4S.	4	0	52	130
Lama glama (L.)	3	8	11	7	7	7	6	5	8	2S.	4	53	10	135
Equus caballus L. (dom.)	2	11	13	10	10	8	9	7	10	3S.	5	33	20	90
Bos taurus L. (dom.)	3	11	14	12	12	10	9	10	11	3S.	5	30	36	100
Giraffa camelopardalis (L.)	2	9	11	6	8	8	8	7	13	2S.	4	70	0	85
Average of running mammals	3	11	14	9	9	8	8	7	10			42	18	110
Hippopotamus amphibius L.	4	14	18	16	12	10	9	8	11	2S.	3	75	0	90
Rhinoceros sondaicus Desm.										2S.	4	70	0	90
Elephas maximus L.										2S.	3	75	0	90
Average of heavy mammals	4	14	18	16	12	10	9	8	11			73	0	90
Trichosurus vulpecula (Kerr)	10	12	22	12	12	9	10	15	10	c.2S.	2	22	0	135
Phascolarctos cinereus (Goldf.)	13	7	20	8	7	8	7	7	7	c.3S.	3	30	0	112
Sciurus vulgaris L.	10	13	23	14	12	7	7	10	10	c.2S.	2	0	20	150
Trachypithecus pyrrhus (Horsf.)	12	10	22	9	12	9	9	8	11	c.2S.	2	10	20	100
Average of climbing mammals	11	11	22	11	11	8	8	10	10			15	10	124
Dendrolagus inustus Müll.u.Schl	14	9	23	15	9	13	13	18	12	c.2S.	2	36	0	130
Bettongia lesueuri grayi Gould	14	10	24	21	14	13	13	20	7	c.2S.	2	22	0	130
Macropus giganteus (Zimm.)	14	8	22	16	16	11	10	15	9	c.2S.	2	22	18	130
Pedetes caffer (Pall.)	17	13	30	19	8	15	15	23	16	c.3S.	3	0	20	115
Jaculus jaculus (L.)	15	11	26	20	17	15	11	20	20	c.3S.	3	0	20	150
Av.of bipedal jumping mammals	15	10	25	18	13	13	12	19	13			16	12	131
Ateleus paniscus (L.)	11	24	35	14	13	12	12	16	24	c.3S.	3	5	23	112
Hylobates lar leuciscus Geoffr.	11	22	33	9	12	13	20	15	26	c.4S.	4	22	16	90
Pongo pygmaeus (Hoppius)	15	23	38	19	15	15	16	19	23	3S.	3	5	35	120
Av. hanging-climbing mammals	12	23	35	14	13	13	16	17	24			11	25	107
Homo sapiens L.	9	12	21	16	8	20	21	19	15	1S.	5	5	85	80

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1) c. 2 S. = caudal of the 2d sacral vertebra.

2) See sub 3.

(To be continued.)

Psychologie. — *Das Problem des Ursprungs der Sprache.* III. Von G. RÉVÉSZ. (Communicated by Prof. A. P. H. A. DE KLEYN.)

(Communicated at the meeting of February 28, 1942.)

G. *Die bewusstseinspsychologische Theorie.*

Man hat das Problem der Entstehung der Sprache auch dadurch zu lösen versucht, dass man die Frage aufwarf, aus welchen *Funktionen des Bewusstseins* oder aus welcher allgemeinen menschlichen *Veranlagung* heraus die Sprache überhaupt möglich war und in welcher Weise diese bei ihrer Entstehung und Ausbildung mitgewirkt haben. Es wird hierbei wieder meistens auf die Ausdrucksbewegungen, Gebärden, Interjektionen als auf die „Rudimente" der menschlichen Sprache hingewiesen, und darin tritt eine Auffassung zutage, deren Unhaltbarkeit wir bereits nachgewiesen haben. Man stellt sich rein logisch-konstruktiv vor, wie aus diesen Rudimenten die Sprache entsteht. „Im Augenblick, in welchem ein bestimmter Lock-, Warn- oder Schreckensruf die Gestalt gewonnen hatte um nicht nur einen Zustand, sondern daneben (!) das erregende Objekt und seine Tätigkeit zu bezeichnen (!), — dieser Augenblick darf die Geburtsstunde der Sprache im Sinne der Gedankenmitteilung genannt werden", sagt JODL in seiner Psychologie[23]). Ja, aber gerade auf das „daneben", auf das „bezeichnen", das ein ganz neues Moment darstellt, kommt es an, also auf die *Objektivierung* des Zustandes, auf die Verwendung des Lauts als eines allgemeinen Symbols, d.h. auf das *Wort*. *Wie* aus dem Schreckens*ruf* einmal ein solches *Wort* entsteht, das wird nicht erklärt. Aehnlich ist es, wenn man aus dem Greifakt das Zeigen oder Begreifen abzuleiten versucht (WUNDT, CASSIRER) und darauf hinweist, dass bei der kindlichen Entwicklung die eine Funktion zeitlich auf die andere folgt. Man fällt einem Irrtum anheim, wenn man diese zeitliche Aufeinanderfolge von spezifischen Tätigkeiten in der geistigen Entfaltung des Individuums als eine *stetige* auffasst. Die zeitliche Aufeinanderfolge von Tätigkeiten — wie gesetzmässig sie auch sein mag — darf nicht ohne weiteres als innere Entwicklung verstanden werden. Das Auftreten des Sprachaktes nach vorangegangener emotionaler Lautäusserung oder der Zeigebewegung, welche die Möglichkeit von Greifakten voraussetzt, sagt über die gegenseitige Beziehung dieser Aktivitäten nichts aus. Die chronologisch aufeinanderfolgenden Aktivitäten sind in ihrem Wesen so verschieden, dass sie nicht als blosse Stufen einer kontinuierlichen Entwicklung zu nehmen sind. Wie voreilig eine solche Theorie ist, wird durch die Erwägung klar, dass das Zeigen und das Hinweisen das Sprachverständnis voraussetzen. Ein Kind, wie experimenteel nachgewiesen worden ist, weist mit dem Finger erst dann auf einen Gegenstand oder auf eine Person, wenn bei ihm die aktive oder wenigstens die passive Sprachfunktion bereits in Wirksamkeit getreten ist.

Man könnte auch anders vorgehen und sagen, dass für die Entstehung der Sprache eine bestimmte seelische oder geistige Konstitution des Menschen die Voraussetzung bildet. Das ist zwar richtig, aber man sollte nicht glauben, dass man mit der Festlegung solcher Bedingungen der Antwort auf die Frage nach dem Ursprung der Sprache näher kommt. Unter den betreffenden Bedingungen wird man nämlich Funktionen antreffen, die, wie z.B. das Denken, Abstrahieren, die Fähigkeit zur Begriffsbildung und dgl. mehr, ohne Sprache nicht vorstellbar sind. Wollte jemand im *Denken* die Grundvoraussetzung der Sprache sehen und die Sprache für ein blosses *Produkt* des Denkens halten, so müsste er zu der Vorstellung eines Menschen kommen, der zwar denkt, aber noch nicht spricht. Das würde eine mit der Erfahrung und mit einer sinnvollen Interpretation tierischer und kindlicher Aeusserungen unvereinbare Vorstellung sein[24]). Aber auch abgesehen davon würden die Schwierigkeiten dadurch nicht aufgehoben werden; es würde bloss an die Stelle des Problems des Ursprungs der Sprache ein neues Problem treten, nämlich das des Ursprungs des Denkens.

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[23]) FR. JODL, Lehrbuch der Psychologie. 1903. II. S. 230.

Wie mit der Sprache, so steht es auch mit der Disposition zur Sprache und zum Sprechen. Eine solche Disposition ist bei allen Menschen und nur bei Menschen vorhanden und sie ist bei ihnen im wesentlichen gleichartig. Von hieraus erklärt es sich, dass alle Menschen — die an Taubheit leidenden einbegriffen — im Prinzip die Fähigkeit besitzen, *alle Sprachen* der Welt zu verstehen und zu bemeistern. Der Sprache im allgemeinen genommen, also allen Sprachen, müssen daher die *gleichen geistigen* Voraussetzungen eigen sein.

H. *Die Bedeutung der Ethnologie und Pathologie für das Ursprungsproblem.*

Der Vollständigkeit wegen will ich noch auf zwei Gesichtspunkte hinweisen, die gelegentlich bei der Behandlung des Ursprungsproblems der Sprache angewendet werden. Auch da hat man den Ursprung mit der Fortentwicklung verwechselt.

Der erste Gesichtspunkt lenkt den Blick auf das ethnologische Material, auf die *primitiven Sprachen*. Man versuchte, die Anfänge der Sprache, mitsam die „Ursprache", aus den sprachlichen Aeusserungen primitiver Völker abzuleiten. Gibt man auch zu, dass aus dem Lautmaterial und der Struktur der primitiven Sprachen mit einiger Wahrscheinlichkeit die Urform der gesprochenen Sprachen zu rekonstruieren ist, so wird man dennoch dieses Prinzip bezüglich der Stufen der vorsprachlichen Periode nicht gelten lassen können. Prinzipiell ist allerdings die Möglichkeit nicht von der Hand zu weisen, dass eine primitive Sprache, die sich gerade in der ersten Periode der Sprachentwicklung befindet, gewisse Aufklärungen über das Laut- und Gebärdenmaterial geben kann, das in der vorsprachlichen Periode der „Menschheit" als Kontaktmittel diente. Diese Möglichkeit müssen wir aber wegen des hohen Alters der primitiven Völker und ihrer Sprachen ausschliessen. Alle gegenwärtig gesprochenen primitiven Sprachen, die die Sprachforscher „Wurzelsprachen" nennen, wie einige sudanesische Sprachen und die Sprachen der Pygmaeen, sind bereits *Sprachen*, welche die wesentlichen Merkmale der Sprache enthalten, so dass sie uns über den hypothetischen Urzustand des sprachlichen Mittels keine Aufklärung zuteil werden lassen können. Die Sprachen primitiver Völker im allgemeinen sind bereits vollkommene Sprachen, zum Teil sogar von entwickelter und komplizierter Form, jedenfalls solche, die die konstitutiven Eigenschaften der Sprache, wie durch Begriffe bestimmte Worte, Sätze, grammatikalische Kategorien und syntaktische Formen, aufweisen. Die Untersuchung der primitiven Sprachen kann uns höchstens von der Entfaltung, von den Anfangsphasen der schon gesprochenen Sprache eine Vorstellung geben, nicht aber von dem Ursprung.

Was die *aphasischen Erscheinungen* betrifft, so handelt es sich hier um einen pathologischen *Abbau des Sprachverständnisses*, also um einen Vorgang, der auf die ganze geistige Struktur des Patienten Einfluss ausübt. Die aphasischen Erscheinungen sind so kompliziert und variabel, dass sie — selbst dann, wenn es sichergestellt wäre, dass nach Analogie dieses *Abbau*prozesses ein entsprechender *Aufbau*prozess rekonstruiert werden könnte — als Modell für die vorbereitenden Etappen der Sprache nicht in Frage kommen. Das aphasische Sprachmaterial, wie lückenhaft und verändert es auch sein mag, gehört zu dem Gebiet der *Sprache*; folglich ist es nicht möglich, es bei der Rekonstruktion eines Zustandes zu verwenden, welcher *vor* der Entstehung der Sprache liegt. —

Alle Bemühungen, in Hinblick auf gewisse scheinbare Parallelen das Ursprungsproblem zu lösen, haben sich als vergeblich herausgestellt. Kinder sind *Menschen*kinder, sie sind konstitutionell auf das Sprechen vorbereitet und mit einem innerem Sprachsinn begabt.

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[24]) G. RÉVÉSZ, Denken, Sprechen und Arbeiten. Archivio di psicologia, neurologia, psichiatria. I. 1940.

Ihre lautlichen und sprachlichen Aeusserungen können keine *Urstufe* der Sprache repräsentieren; denn im Keime ist bei ihnen bereits eine *hochentwickelte* vererbte Sprachfähigkeit vorhanden, die früh einsetzt und sich erstaunlich schnell entwickelt. Die Sprachen der Primitiven sind *Sprachen,* die eine Entwicklung von Tausenden von Jahren hinter sich haben. Die Annahme endlich, dass man in den *Sprachstörungen* einen Hinweis auf die Uranfänge der Sprache finden könne, stützt sich auf ein empirisches Material, das keine Anknüpfungspunkte für die vorsprachliche Periode liefert.

4. *Das Problem der Ursprache.*

Im Zusammenhang mit dem Problem des Ursprungs der Sprache treten noch zwei Fragen auf, die ich noch kurz berühren möchte. Die erste ist die Frage nach der *Ursprache.*

Die Idee eines Stammbaumes der Sprache ist in der neueren Sprachforschung aufgegeben worden. Bereits JOH. SCHMIDT hat das Phantom einer Ursprache vernichtet[25]). Wie *der* Urmensch, *das* Urvolk, so ist *die* Ursprache ein Phantasiegebilde, eine nicht fundierte Hypothese, die infolge von einer unberechtigten Interpretation des Entwicklungsbegriffes entstanden ist.

Die Ursprache soll eine Sprache sein, deren sich die Menschen in der Urzeit bedienten, als sie gerade „anfingen" die Lautsprache zur Mitteilungszwecken zu gebrauchen. Für eine derartige Hypothese fehlen uns vollkommen die historischen, ethnologischen und vergleichend-sprachwissenschaftlichen Belege. Will man die Idee der Ursprache, d.h. der primitivsten Form der Sprache, nicht aufgeben, so würde es sinnvoller sein, von *mehreren* Ursprachen zu reden, von der indoeuropäischen, semitischen, malayischen usw., d.h. von Sprachen, die voneinander nicht abzuleiten sind. Die Annahme einer einzigen Ursprache setzt voraus, dass ein bestimmter, geographisch zusammenhängender Teil der Erde als Urheimat der Menschen angesehen wird. Diese Urheimat ist der Ansicht von zahlreichen Paläonthologen und Prähistorikern zufolge Zentralasien; von da aus sollen sich die Menschen über die ganze Erdoberfläche zerstreut haben, wie dies ARLDT in seiner Anthropogeographie darzustellen versucht hat[26]). Sieht man noch ganz ab von den geographischen Schwierigkeiten, die gegen eine solche Anschauung sprechen, so will diese auch mit den paläonthologischen Erfahrungen nicht übereinstimmen. Es ist nämlich bekannt geworden, dass Reste von Menschen aus der frühen prähistorischen Zeit in allen Weltteilen zu finden sind. Dazu kommt noch, dass auch von den diluvialen Hominiden, — die als Uebergangsformen zwischen den diluvialen Anthropomorphen und dem rezenten Menschen betrachtet werden — beinahe in allen Teilen der Erde fossile Reste gefunden werden. Will man die Hypothese der Urheimat und der Ursprache einigermassen plausibel gestalten, so muss man den Zeitpunkt der angeblichen prähistorischen Wanderungen in die ferne diluviale Zeit verlegen, vor allem aber es wahrscheinlich machen, dass die Hominiden, die an diesen Wanderungen teilgenommen haben, sich bereits der Sprache, der angeblichen Ursprache, bedienten. Meiner Meinung nach wird die Rekonstruktion einer Ursprache auch unter diesen Voraussetzungen keinen Anspruch auf wissenschaftlichen Wert erheben können. Das Problem der Ursprache der Menschheit erweist sich nicht bloss als unlösbar, sondern geradezu als widersinnig[27]).

5. *Die Sprache des Urmenschen.*

Anders steht es mit der zweiten Frage, nämlich die, ob die Sprachfunktion nur bei dem rezenten Menschen anzunehmen ist oder ob man berechtigt ist, den sog. Vorfahren des rezenten Menschen, wie dem *homo neanderthalensis* oder mindestens den frühesten

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[25]) FR. MAUTHNER, Kritik der Sprache. 1912. Bd. II. S. 375.

[26]) TH. ARLDT, Die Entwicklung der Kontinente und ihrer Lebewelt. 1907. Vergl. dazu A. J. P. VAN DEN BROEK, Oudste Geschiedenis van den Mensch. 1936.

[27]) Eingehend behandelt diese Frage und die mutmassliche Beschaffenheit der Urlautgebilde F. KAINZ in seinem Werk „Psychologie der Sprache". I. Leipzig 1941.

Formen des sog. homo sapiens, die den Namen Aurignac-Mensch, Crô-Magnon-Mensch etz. führen, die Sprachfunktion zuzuerkennen.

Von biologischer Seite hat man versucht, diese Frage auf vergleichend-anatomischem Wege zu beantworten. Dass keiner von diesen Versuchen zu einem befriedigendem Resultat führte, war eigentlich zu erwarten. Der Hinweis auf die grosse Kapazität des Schädels, mithin auf das zu vermutende grosse Gewicht des Gehirns des Neanderthalers, ist irreführend (KEITH, BOLK, ANTHONY). Schädelgrösse kann wohl als eine der Vorbedingungen für die Sprachtätigkeit gelten; sie bietet aber für das Vorhandensein dieser keine Gewähr, noch ganz abgesehen davon, dass wir von der anatomisch-morphologischen Struktur des Gehirns dieser Fossilen nicht die geringste Kenntnis haben. Auch WEINERT lässt sich durch den Körper- und Schädelbau des Neanderthalers imponieren. Beträchtlichen Wert hat für ihn die Tatsache, dass der Gehirnraum des Neanderthalers im Durchschnitt ebenso gross ist wie der unsrige. Obgleich er einerseits davor warnt, hieraus Schlüsse auf das Intelligenzniveau zu ziehen, geht er andererseits so weit, ohne trifftige Gründe dem Neanderthaler Sprach- und Denkfähigkeit zuzuschreiben [28]).

Die also auf solch eine vergleichende Anatomie sich stützenden Ueberlegungen bezüglich der Sprache sind irreführend. Sie ebnen den Weg zu Phantastereien und lassen die diluvialen Hominiden viel entwickelter erscheinen als sie es in Wirklichkeit gewesen sind. So kommt man denn dazu, die vormenschlichen Typen als ausgestorbene Menschenformen in die Systematik der heute lebenden Menschen einzureihen. Das zeigt sich in eklatanter Weise bei ELLIOT SMITH [29]) und KEITH [30]), die sogar gegenüber den kritischen Ausführungen von DUBOIS [31]) selbst den Pithecanthropus erectus Sprachvermögen zuerteilen.

Die Frage nach der Sprachtätigkeit längst ausgestorbener primitiver Menschenarten lässt sich meiner Meinung nach nur von einem einzigen Standpunkt aus in Angriff nehmen und das ist der *kulturhistorische.*

In diesem Falle verzichtet man auf den Versuch festzustellen, ob der diluviale Mensch gesprochen hat oder nicht; denn diese Frage lässt sich nicht beantworten. Man trachtet, das Problem indirekt zu lösen, indem man prüft, ob bei diesen Menschenarten gewisse Voraussetzungen erfüllt waren, die mit der Existenz der Sprache *notwendig* verbunden sind. Eine von diesen Voraussetzungen ist die *Verfertigung* und allmähliche *Verbesserung* von Werkzeugen und Geräten zur Ausführung von Arbeitsleistungen, ferner ihre *Ueberlieferung.* Gelingt es uns zu konstatieren, dass der Mensch des Jungpaläolithikum Werkzeuge herstellte, überlieferte und allmählich fortentwickelte, so war er ein Mensch im eigentlichen Sinne, also ein homo sapiens, der als solcher die Sprache besass; denn das Finden und Herstellen von stabilen, der Arbeit adäquaten, durch Absicht entstandenen und zielgerichteten Werkzeugen setzt die Sprache voraus. Homo *faber* lässt sich meiner Ansicht nach mit homo *loquens* identifizieren [32]). Es kommt also darauf an, ob die entdeckten fossilen Reste und die in denselben Ablagerungen gefundenen Kulturreste zueinander gehören oder nicht. Ist das der Fall, dann sind wir berechtigt, bei den betreffenden Menschenarten die Sprachfunktion vorauszusetzen. Da die gestellte Bedingung in Hinblick auf die jungpaläolithischen Menschen erfüllt ist, kann man mit Wahrscheinlichkeit annehmen, dass sie bereits die Sprache hatten. Damit setzen wir den Anfang der Sprache für eine Zeit an, die 50.000 bis 100.000 Jahre zurückliegt.

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[28]) H. WEINERT, Menschen der Vorzeit. 1930.

[29]) E. SMITH, The Searche for man's ancestors. 1931.

[30]) A. KEITH, The antiquity of man. 1920.

[31]) E. DUBOIS, De beteekenis der groote schedelcapaciteit van Pithecanthropus erectus. Proc. Kon. Akad. v. Wetenschappen, Amsterdam. 1920.

[32]) Diese meine Behauptung habe ich in der Abhandlung zu beweisen versucht, in der ich die Beziehung zwischen Sprache und Arbeit auseinandergesetzt habe (zitiert oben Fussnote 24).

6. *Prinzipielle Bedenken gegenüber der Ursprungstheorien.*

Aus den vorangehenden Erörterungen hat sich ergeben, dass die bisher inbetreff des Ursprungs der Sprache aufgestellten Theorien unhaltbar sind. Keine von ihnen ist theoretisch und empirisch zureichend fundiert, und keine ist imstande, die Lücke zwischen einem vorsprachlichen und vollsprachlichen Stadium zu überbrücken, den Spracherwerb als Endpunkt einer allmählichen Entwicklung darzustellen.

Werfen wir nun die Frage auf, warum die bisherigen Theorien über den Ursprung, über die Vorstufen der Sprache so ergebnislos geblieben sind, so finden wir, dass sie Irrtümer enthalten, die zu den Ursachen dafür gehören, dass man vergebens nach einer befriedigenden Antwort suchte.

Auf die Zweideutigkeit der Problemstellung und auf die Nichtbeachtung prinzipieller Voraussetzungen der Hypothesen haben wir schon in Kap. 2 hingewiesen. Darauf brauchen wir nicht mehr einzugehen.

Weitere Mängel liegen vor, sowohl was die *methodologischen* wie auch was die *inhaltlichen* Gesichtspunkte betrifft.

Zunächst versäumte man es einen auf eine genaue Analyse gegründeten *Begriff der Sprache* zu geben. Man hätte wissen müssen, dass zunächst einmal Klarheit darüber geschaffen werden muss, was man unter Sprache verstehen will. Erst dann, wenn man sich von der Sprache eine wissenschaftlich berechtigte Vorstellung gebildet hat, wird man an die Aufgabe herantreten können, Gedanken über die mutmasslichen Vorstufen der Sprache zu entwickeln. Eine klare, wenn auch nur vorläufige Begriffsbestimmung der Sprache ist vor allem erforderlich, um zu entscheiden, was man zu den Vorstufen und was man zu den eigentlichen sprachlichen Aeusserungen zu rechnen hat. Definieren wir die Sprache als die Funktion, durch die wir mit Hilfe einer Anzahl von gegliederten und in verschiedenen Sinnverbindungen auftretenden Laut- bezw. Bewegungs- oder Zeichengebilden unsere Wahrnehmungen, Urteile, Wünsche darzustellen und in der Absicht gegenseitiger Verständigung anderen mitzuteilen imstande sind, dann kommen wir nicht in Verlegenheit bei der Entscheidung, was als Sprache, was als sprachlose Kommunikationsform und was nur als Vorstufe der Sprache zu betrachten ist[33]).

Der geringen Sorgfalt, welche die Ursprungsforscher bei der Analyse und Begriffsbestimmung der Sprache an den Tag legten, ist es zuzuschreiben, dass man die Sprache auf Funktionen zurückführte, die bereits die Sprache voraussetzen, wie z.B. die Gebärde und die Sprachäusserungen der Primitiven, oder auf solche, die zu der Sprache in keiner Beziehung stehen, wie z.B. die Lautimitationen.

Zweitens haben meiner Ansicht nach die meisten Sprachtheoretiker sich selbst den Zugang zu der Erforschung der Ursprungsfrage dadurch versperrt, dass sie ein sekundäres Merkmal in den Vordergrund schoben, nämlich das *Medium* der Sprache, den *Laut* und die *Bewegung.* Ihre ganze Aufmerksamkeit wurde von dem *Mittel* und nicht von der *treibenden und bildenden Kraft in Anspruch genommen.* Da in der menschlichen Sprache der Stimmlaut der Ausdrucksmittel par excellence ist, ist es begreiflich, dass man die Sprache auf spontanen Lautäusserungen zurückzuführen suchte, zumal die letzteren in ihrer Erscheinungsweise und ihrem lautlichen Charakter Aehnlichkeiten mit den Sprachlauten aufweisen. So kam es zur Theorie der emotionellen Lautäusserungen und Interjektionen und auch zu der der Onomatopoeia. Die Naturlaute sollen die Vorstufe und zugleich den phonetischen Grundstock der Sprache gebildet haben. Man hat hierbei irrtümlicherweise die Entwicklung der Sprache mit ihrer Entstehung verwechselt; den Theoretikern ist es entgangen, dass Ausdrucks- und Nachahmungslaute wohl bei der *Entwicklung* der Sprache eine Rolle spielen können, nicht aber bei ihrer *Entstehung.* Sie gewinnen erst Bedeutung, wenn die Sprachtätigkeit bereits eingesetzt hat, wenn der Mensch im Verkehr mit seinen Artgenossen nach Wortlauten sucht. Dasselbe wird wohl auch für die Ausdrucksbewegungen gelten. Ihre Transformation zu Gebärden setzt erst im Verlauf der Entwicklung der Sprache ein, vermutlich schon in der Frühperiode der Sprache, als nämlich die ersten Worte, Sprachsymbole und sprachlichen Formen mit Gebärdensymbolen zusammen gleichzeitig auftreten.

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[33]) Eine definitorische Festlegung wird immer erforderlich sein, wenn man Vermutungen über die Vorgeschichte von menschlichen Tätigkeiten anstellen will. So muss man auch bei der Frage nach dem Ursprung der Musik von vornherein wissen, was man unter Musik zu verstehen hat. Sehen wir schon in der monotonen Klangerzeugung die ersten Regungen der Musik, dann werden wir die monotonen Trommelschläge exotischer Völker zur Musik rechnen. Wollen wir indessen erst da von Musik reden, wo feste Intervalle und ihre Kombinationen auftreten, dann müssen diese monotonen Trommeltöne, sowie die Lallmelodien kleiner Kinder und auch der sogenannte Vogelgesang aus der Betrachtung ausgeschaltet werden. (Siehe meine Betrachtungen in meiner Schrift über den Ursprung der Musik im Intern. Archiv. f. Ethnographie, Bd. 40, 1941).

Aus diesen Ueberlegungen wird ersichtlich, dass eine vorläufige Definition unter Umständen nicht nur nicht überflüssig, sondern geradezu notwendig ist. Dies zeigt sich auch bei der Frage nach dem Ursprung der bildenden Kunst. Auch hier wird die Hypothesenbildung davon abhängen, ob man gewisse Formen und Funde menschlicher Arbeit auch ohne Nachweis des Kunstwollens, als „Aeusserung der Kunst" gelten lässt oder nur solche, die deutlich von künstlerischen Absichten bestimmt sind.

Wie bedeutungsvoll also auch das Stoffliche für die Entwicklung der Sprache sein mag, bei ihrer Entstehung kann es nur eine untergeordnete Rolle gespielt haben. Das Vorhandensein des Mediums ist allerdings notwendig, analog der Hand für die menschliche Arbeit. Wie aber unser Greiforgan als solches ohne treibende Kraft zu keiner Arbeit im stande ist, wie dies bei anthropomorphen Affen zu sehen ist[34]), so kann auch der Laut aus eigener Kraft nicht zum Mittel der Sprache werden. Das zeigen die stimmgebenden Vögel und die lallenden jedoch sprachunfähigen Idioten auf evidente Weise. Es ist also keineswegs überraschend, dass auf das Stoffliche gerichtete Hypothesen bei der Rekonstruktion der vorsprachlichen Etappen für die Forschung keine fruchtbare Gesichtspunkte zu liefern imstande waren.

Das Wesentliche der Sprache liegt nicht in den äusseren *Mitteln,* mit deren Hilfe die Gedanken ihre Verkörperung finden, sondern im *Zweck.* Vom teleologischen Standpunkt aus ist die Sprache eine *Kommunikationsform* von reichster Gestaltung. Will man von ihrem allmählichen Zustandekommen eine Vorstellung gewinnen, die Vorgeschichte der Sprache gleichsam rekonstruieren, so muss man von jenen *Kommunikationsformen* ausgehen, die im vorsprachlichen Stadium vermutlich den Kontakt zwischen den vormenschlichen Wesen gebildet haben. Hierbei können nur solche Kommunikationsformen in Betracht kommen, die von *demselben allgemeinen Prinzip* beherrscht werden wie die Sprache.

Die Sprachtheoretiker haben die Wichtigkeit eines Grundprinzips, das gleichsam das Bindeglied zwischen Sprache und vorsprachlichen Aeusserungen zu bilden hat, nicht erkannt. Sie meinten, der Entwicklungsidee genüge zu leisten durch den Hinweis auf gewisse Lebensäusserungen, die beim Menschen vorliegen und auch bei unseren hypothetischen Vorfahren anzunehmen sind. Sie haben nicht bemerkt, dass die Lebensäusserungen, die sie in den affektiven Lautäusserungen und Ausdrucksbewegungen zu finden meinten, mit der Sprache als solche nichts gemein haben und niemals zur Sprache führen konnten, einfach aus dem Grunde, weil sie anderen Zwecken dienen, andere Wurzel haben und andere Prinzipien unterworfen sind.

Aus diesen Ueberlegungen folgt, dass man nach einem Prinzip suchen muss, welche alle Kommunikationsformen, einschliesslich der Sprache, in ihrem Zustandekommen und ihrer Funktion bestimmt, und aus diesem allgemeinen Prinzip muss versucht werden das spezifische Prinzip abzuleiten, welches die entwickelteste Kommunikationsform, die Sprache, in ihrem Wesen bestimmt. In der Aufstellung dieser Prinzipien liegt der Grundgedanke der hier geschilderten *Sprachtheorie.*

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[34]) G. RÉVÉSZ, La fonction sociologique de la main humaine et de la main animale. Journ. de Psychologie, 1938. Ferner: Die Formenwelt des Tastsinnes, Den Haag, Band I, 1937.

Comparative Physiology. — *Das p_H-Optimum der Darmmaltase beim Schweine.* Von L. M. VAN NIEUWENHOVEN S. J., D. P. NOORDMANS und H. J. VONK. (Aus dem Laboratorium für vergleichende Physiologie der Universität Utrecht). (Communicated by Prof. H. J. JORDAN).

(Communicated at the meeting of February 28, 1942.)

Die proteolytischen Enzyme des Säugerdarmes sind wiederholt Gegenstand wissenschaftlicher Untersuchung gewesen. Viel weniger hat sich das Interesse der Untersucher dem Studium der disaccharidspaltenden Fermente des Darmes zugewendet. Von den Disaccharasen wurden vorwiegend diejenigen der Hefe und anderer niederer Pflanzen untersucht.

Besonders wenig hat man sich um die Maltase des Darmes gekümmert. Während das p_H-Optimum aller anderen bekannten Enzyme des Verdauungstraktus der Säugetiere festgestellt worden ist, und sogar für verschiedene Enzyme vielfach und eingehend untersucht wurde, fehlt merkwürdigerweise eine derartige Bestimmung für die Darmmaltase der Säugetiere. Es ist dies um so bemerkenswerter, da die Maltase vom Standpunkte der Verdauungsphysiologie betrachtet, wohl die wichtigste Carbohydrase des Darmes ist. Beendet sie doch die Spaltung von Stärke und Glykogen, welche vom Pankreassaft, und bei einigen Säugern auch von der Speichelamylase, eingeleitet wird.

Die Feststellung des p_H-Optimums für dieses Enzym ist wichtig für die Entscheidung der Frage, ob die Reaktion des Darminhaltes der Maltase nahezu optimale Wirkungsbedingungen sichert oder nicht. Die biologische Bedeutung des p_H-Optimums der Verdauungsenzyme bei den Vertebraten wurde von einem von uns [1]) eingehend erläutert, so dass wir für eine ausführliche Besprechung dieser Frage auf diese Arbeit verweisen können.

Bei niederen Vertebraten wurde das Optimum der Maltasewirkung schon bestimmt [2]). Für das Pankreas des Karpfens liegt es zwischen p_H 6.6 und 7.1, für die Darmwand von Testudo graeca bei 7. Bei Invertebraten liegt das Optimum dieses Enzyms niedriger; so fanden WIERSMA und VAN DER VEEN [3]) sowie KRÜGER und GRAETZ [4]) es für Astacus bei 5.3—6.0. In all diesen Fällen ist die Optimumkurve sehr flach.

Weiter sei noch erwähnt, dass EULER und SVANBERG [5]) das Optimum für die Darmsaccharase bestimmten. Die von ihnen gefundene Optimumkurve ist ebenfalls sehr flach. (Optimalzone 5—7).

Obgleich wohl kein überraschendes Resultat für das p_H-Optimum der Darmmaltase der Säugetiere zu erwarten war, schien es uns nützlich es zu bestimmen, da das Fehlen dieser Bestimmung doch eine gewisse Lücke in unserer Kenntnis der Verdauungsenzyme bedeutet. Vorläufig mussten wir uns begnügen mit der Feststellung des Optimums für Darmextrakt, da Fistelsaft uns nicht zur Verfügung stand. Nach UNGER [6]) sind Schweine mit Darmfisteln sehr schwierige Versuchsobjekte und der Hund war als Fleischfresser für diese Versuche ungeeignet.

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[1]) H. J. VONK, Ergebnisse der Enzymforschung **8**, 55 (1939).

[2]) H. J. VONK, Zs. vgl. Phys. **5**, 445 (1927).
H. P. WOLVEKAMP, Zs. vgl. Phys. **7**, 454 (1928).

[3]) C. A. G. WIERSMA und R. VAN DER VEEN, Zs. vgl. Phys. **7**, 269 (1928).

[4]) Siehe P. KRÜGER, S. B. Preuss. Akad. Wiss. **26**, 1 (1929).

[5]) H. VON EULER und O. SVANBERG, Zs. physiol. Chem. **115**, 43 (1921).

[6]) H. UNGER, Inauguraldissertation Hannover 1938.

Material und Methodik. Der Enzymextrakt wurde folgenderweise erhalten. Vier oder sechs Schweinsdärme wurden mit Wasser gut ausgespült, aufgeschnitten und die Schleimhaut mittels eines Objektglases vorsichtig von der Muskularis abgekratzt. Die gesammelte Schleimhaut wurde gewogen und mit der zweifachen Menge Glyzerin und mit ausgeglühtem, gewaschenem und nachher getrocknetem Quarzsand verrieben. Zur besseren Zerquetschung der Gewebsteile geschah das Verreiben mit einer kleinen Menge Glyzerin und wurde nachher erst der Rest zugesetzt. Nach 4-bis 7-tägigem Stehen an einem kühlen Orte wurde die Masse durch ein Kolliertuch filtriert oder darin ausgepresst. War der Extrakt noch nicht genügend klar, so wurde noch zentrifugiert. Auf diese Weise wurden zwei Extrakte (A und B) erhalten. Für die Enzymversuche wurde eine bestimmte Menge des Extraktes mit vier Volumina Wasser verdünnt. Hierbei entstand eine leichte Trübung, welche abzentrifugiert wurde. Von der klaren Lösung wurden dann 10 cm³ in die Erlenmeyerkolben, welche Maltose und Puffergemisch enthielten, pipettiert.

Mit dem Darmextrakt A wurde das p_H-Optimum bestimmt mittels der Zuckertitration von BERTRAND (modifiziert nach SCHOORL). Für die Zuckertitration haben wir die Flüssigkeiten nicht enteiweisst, da die vom Extrakt herrührende Eiweissmenge nur sehr gering war. Statt der vorgeschriebenen mit Asbestwolle beschickten perforierten Tiegel, haben wir zur Vereinfachung versucht, für das Filtrieren einen B 2 Tiegel zu benutzen. Wenn kein Eiweiss vorhanden war, gab diese Anordnung sehr gute Resultate. Bei den eigentlichen Versuchen mit verdünntem Darmextrakt aber, filtrierte die Flüssigkeit wegen des Eiweissgehaltes sehr langsam, so dass wir für diese Bestimmungen wieder den mit Asbest bekleideten perforierten Tiegel benutzen mussten.

Mit dem Extrakte B bestimmten wir das p_H-Optimum mit dem Polarimeter. In beiden Fällen wurden die p_H-Bestimmungen elektrometrisch ausgeführt. Als Puffer diente das Veronal-Natriumazetatgemisch nach MICHAELIS, das mit $\frac{N}{10}$HCl auf den erwünschten p_H gebracht wurde[1]. Es stellte sich nämlich bald heraus, dass das p_H-Optimum der Darmmaltase sehr flach und breit war. Es war daher nicht möglich unter Anwendung von Phosphatpuffer den Abfall der Optimumkurve nach der sauren und alkalischen Seite zuverlässig festzustellen.

Es wäre möglich, dass der Veronalnatriumpuffer im Vergleich zum vielfach benutzten Phosphatpuffer einen hemmenden oder aktivierenden Einfluss auf die Maltasewirkung ausüben könnte. Daher wurde zuerst ein Parallellversuch angesetzt mit beiden Puffern bei ungefähr gleichem p_H.

Ansatz: 13 cm³ Pufferlösung; 12 cm³ dest. Wasser; 25 cm³ 1 % Maltoselösung; 10 cm³ Enzymlösung (Glyzerinextrakt A, 1 : 4 mit Wasser verdünnt und zentrifugiert).

Resultat:

	Veronalpuffer p_H 7.13		Phosphatpuffer p_H 6.85	
	cm³ $KMnO_4$[2]	Zunahme	cm³ $KMnO_4$	Zunahme
Blanko	3.98		4.03	
Nach 15 Min.	4.83	0.85	4.80	0.77
„ 30 „	5.33	1.35	5.31	1.28
„ 45 „	5.73	1.75	5.59	1.56
„ 60 „	5.86	1.88	5.94	1.91

Aus diesem Versuch erhellt, dass bei beiden Puffern die Zunahmen der Reduktion, welche von der Enzymwirkung verursacht werden, nur unwesentlich und innerhalb der Fehlergrenzen von einander abweichen. Zwar fiel der Unterschied zwischen beiden

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[1] Zusammensetzung S. bei P. RONA, Fermentmethoden, 2te Aufl. Berlin 1931, S. 66.

[2] 0.1561 N.

p_H-Werten, welchen wir möglichst klein zu machen wünschten, durch zufällige Umstände reichlich gross aus. Aber aus den Optimumkurven wird man später sehen, dass der Unterschied von 0.3 im p_H in diesem optimalen Gebiet unwesentlich ist, sodass wir keinen Grund hatten diesen Versuch zu wiederholen.

Wir wollen darauf hinweisen, dass bei dieser Versuchsanordnung der prozentuale Umsatz in einfacher Weise berechnet werden kann. Für eine kleine Zuckermenge stimmt nach der SCHOORLschen Tabelle 1 cm³ $\frac{N}{10}$ $KMnO_4$ oder $\frac{N}{10}$ Natriumthiosulfat überein mit 5.7 mg Maltose oder 3.15 mg Glukose [1]. Wenn also 1 mg Maltose in Glukose umgesetzt wird, verschwindet ein Reduktionsvermögen das übereinstimmt mit $\frac{1}{5.7}$ cm³ Permanganat (oder Thio) und es tritt ein solches von $\frac{36}{34.2} \times \frac{1}{3.15}$ cm³ an seine Stelle. (Der Faktor $\frac{36}{34.2}$ rührt daher, dass bei der Hydrolyse aus je 342 g Maltose (1 Mol) 360 g Glukose (2 Mole) entstehen. Die Reduktionszunahme bei Umsatz von 1 mg Maltose in 1 mg Glukose beträgt also $\frac{36}{34.2 \times 3.15} - \frac{1}{5.7} = 0.159$ cm³. Dividiert man also die (in cm³ 0.1 N $KMnO_4$) gefundene Zunahme durch 0.159, so erhält man die umgesetzte Maltosemenge in mg und kann daraus den prozentualen Umsatz errechnen. Bei der Bestimmung des p_H-Optimums ist es notwendig, diesen Umsatz bestimmen zu können, damit man sehen kann, ob die Reaktion in der optimalen Zone nicht zu weit vorgeschritten ist. Dieses würde den Unterschied in der Wirkung bei den verschiedenen p_H-Werten natürlich verwischen.

Da der Glyzerinextrakt ziemlich lange Zeit aufbewahrt werden musste, haben wir noch einen Versuch angesetzt, um zu ermitteln, ob während dieser Zeit der Extrakt einen Teil seiner Wirksamkeit eingebüsst hatte. Auch dieser Versuch wurde mit der Zuckertitration ausgeführt. Es ergab sich, dass nach 3 Monaten die Maltasewirkung des Extraktes sich nicht geändert hatte.

Da die Ansätze leicht getrübt waren, war es für die polarimetrischen Bestimmungen notwendig, das Eiweiss zu entfernen. Dies geschah durch Zusatz von vier Volumina Alkohol auf 1 Volumen der Versuchslösung (25 cm³). Nachdem die Flüssigkeit übernacht im Eisschrank gestanden hatte, wurde sie abfiltriert und eingeengt. Letzteres geschah durch Aufblasen eines Luftstromes, oftmals kombiniert mit Erwärmen (nicht über 40°, da sonst leicht Braunfärbung eintritt). Der Rückstand wurde in destilliertem Wasser gelöst, 1 cm³ 10 % Natriumbicarbonatlösung zugesetzt zur Beseitigung eventueller Mutarotation, und in 25 cm³ Messkolben bis zur Marke beigefüllt. Diese Lösungen wurden nochmals durch einen Doppelfilter filtriert um eine leichte Trübung, welche von einer winzigen Fettmenge verursacht wurde, zu entfernen. Sodann wurde die Zuckerkonzentration polarimetrisch bestimmt und mit derjenigen zu Anfang des Versuchs verglichen. Auch hier kann man in ähnlicher Weise, wie bei der Zuckertritration beschrieben wurde, durch Vergleichung der Drehungswinkel von Maltose und Glukose den Umsatz in Prozenten bestimmen.

Die Zusammensetzung der Ansätze für die polarimetrische Bestimmung des p_H-Optimums war die gleiche wie für die titrimetrische. Die p_H-Bestimmungen geschahen bei der titrimetrischen Bestimmung mit der Wasserstoffelektrode, bei der polarimetrischen Bestimmung mit der Glaselektrode (COLEMAN-Apparat).

Resultate. Als Beispiel für die erhaltenen Resultate bringen wir die Kurven der Figur 1 (titrimetrisch) und der Figur 2 (polarimetrisch). Die erstgenannten Kurven beziehen sich auf den Glyzerinextrakt A und auf einen Extrakt der mit einer 0.9 % NaCl-

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[1] Für Glukose wurde der Mittelwert der Reduktion für die ersten 4 cm³ genommen. Die Zunahme der Reduktion ist der Zuckermenge nicht genau proportional.

Lösung von der Darmschleimhaut angefertigt wurde. Man sieht eine Optimalzone zwischen p_H 5.4 und 7.6. Das Optimum ist also sehr flach und breit. Durch die grosse Breite des Optimums konnte die ganze Kurve für den Glyzerinextrakt nicht in einem Versuch bestimmt werden. Die beiden Kurven I und II ergänzen einander also. Die Wirkungen bei diesen beiden Kurven sind einander in der Optimalzone nicht genau gleich. Dies ist wahrscheinlich eine Folge der schwierigen Abmessung des Glyzerinextraktes, so dass in beiden Versuchen die angewandten wässerigen Verdünnungen des Extraktes A nicht genau gleiche Wirkung zeigen könnten. Doch ergänzen die beiden Kurven einander vorzüglich. Die Kurve der polarimetrischen Bestimmungen zeigt eine Optimumzone von 5.0 bis etwa 7.0. Zwischen 5.0 und 5.4 ist der Unterschied zur titrimetrischen Kurve nur gering, von 7.0 bis 7.6 zeigt sie, verglichen mit dieser, einen beträchtlichen Unterschied, da die polarimetrische Kurve dort ziemlich steil abfällt. Auch in anderen, hier nicht abgebildeten Versuchen mit der polarimetrischen Methode zeigte sich dieser Unterschied. Da nun die Bestimmungen mit beiden Methoden mit verschiedenen Extrakten ausgeführt wurden, ist es wahrscheinlich, dass Begleitstoffe für diesen Unterschied verantwortlich sind. Jedenfalls ist der Unterschied vom biologischen Standpunkte betrachtet, nicht erheblich. Nach unserem gegenwärtigen Wissen schwankt der p_H des

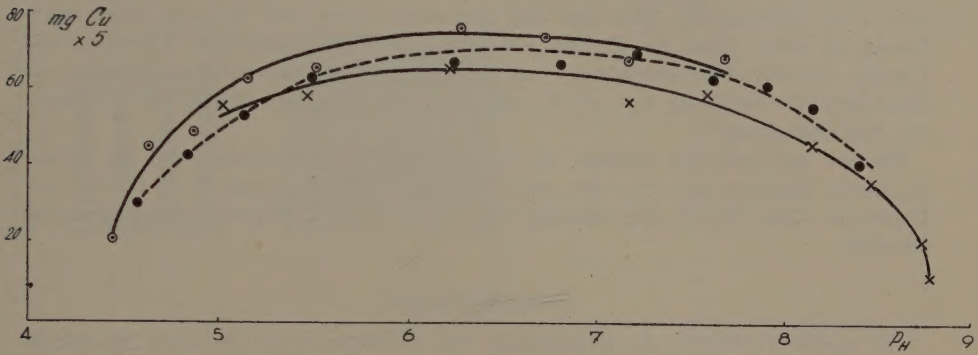

Fig. 1. p_H-Optimum der Darmmaltase vom Schwein, bestimmt mit Zuckertitration. Die Punkte ⊙ und ×, mit ausgezogener Linie verbunden, sind Resultate zweier einander ergänzenden Versuche mit Glyzerinextrakt. Die durch die Punkte ● verlaufende gestrichelte Kurve ist eine Bestimmung mittels NaCl-Extrakt der Darmschleimhaut. Optimalzone zwischen p_H 5.4 und 7.6 (Maximaler Umsatz in diesen Kurven 21.9, bzw. 24.8, bzw. 26.5 %).

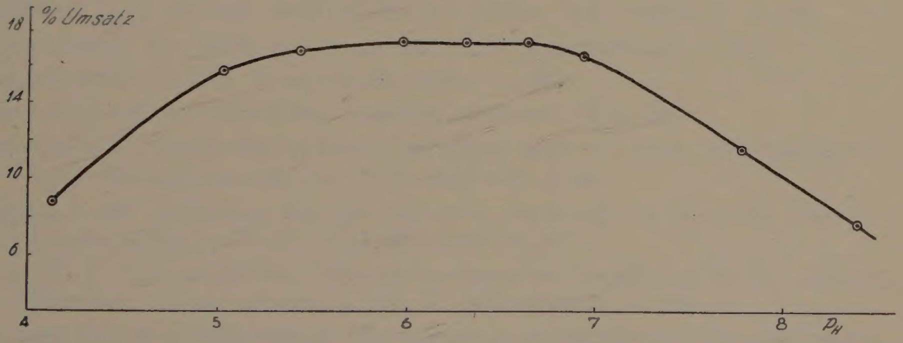

Fig. 2. p_H-Optimum der Darmmaltase vom Schwein, polarimetrisch bestimmt. Optimalzone zwischen p_H 5 und 7.

Darminhaltes für Omnivoren um den Neutralpunkt. Für gefütterte Schweine wurden von LONG und FENGER [1]) in vielen Versuchen als Maximalwert 7.40, als Minimum 6.39 gefunden. (Bei Fleischfressern ist der Darminhalt etwas saurer, bei Pflanzenfressern etwas alkalischer.) Vergleichen wir diese Zahlen mit der für die Darmmaltase des Schweines gefundenen Optimalzone, so können wir schliessen, dass die Maltase unter annährend optimalen Bedingungen wirkt. Dies ist ebenso der Fall für die Amylase, welche bei etwa 7 ihr Optimum hat. Man hüte sich aber davor hierin eine besonders feine Regulierung zu sehen. Denn die Optima von allen daraufhin untersuchten Proteasen und Lipasen weichen nicht unerheblich vom mittleren p_H des Darminhaltes ab. Für die Besonderheiten verweisen wir auf die schon zitierten Zusammenfassung in Ergebnisse der Enzymforschung [2]).

Mit der titrimetrischen Methode wurde ebenfalls das Wirkungsoptimum der Darmsaccharase bestimmt und zwar für einen Glyzerinextrakt und einen NaCl-Extrakt der Darmwand. Diese Bestimmung bestätigte die von EULER und SVANBERG 1921 erhaltenen Resultate.

Zusammenfassung.

Die p_H-Aktivitätskurve der Darmmaltase vom Schweine wurde bestimmt. Sie besteht aus eine breite und flache Optimalzone zwischen p_H 5.2 und 7.2 ungefähr. Beiderseits der Optimalzone fällt die Kurve ziemlich steil ab. Die Darmmaltase des Schweines wirkt im Darminhalt unter optimaler p_H-Bedingung.

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[1]) J. H. LONG und F. FENGER, Jnl. Amer. Chem. Soc. 39, 1278 (1917).

[2]) Fussnote 1 S. 302.

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